Bayesian Methods in Positioning Applications

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***Abstract*— This report presents an outline of Bayesian methods and in particular particle filtering in positioning applications. In the introductory part the general problem of time-variant state-space estimation is reviewed. Bayesian estimation is presented as one of the approaches to tackle this problem and particle filters are introduced as a numerical tool for the calculation of the Bayesian estimator. Throughout this report suitability of the presented methods is discussed with regard to position applications.**

I. Introduction

Particle filters provide a powerful numerical tool for non-linear and non-Gaussian state estimation problems. It is known that for a linear system model, combined with zero-mean Gaussian errors in the observation model the Kalman Filter presents an optimal solution in the MMSE sense [1].

Position determination in localization and tracking applications is inherently a nonlinear problem due to nonlinear mapping of the observed (measured) quantities (e.g. estimated distances $\mathbf{d}$ or differential distances $\delta\mathbf{d}$) to the system state (which is in the simplest case the unknown position $\mathbf{p}$). In radiolocation, measurement imperfections originate from two sources. Properties of the propagation channel (signal fading, bias) put a physical limit on the estimation accuracy. Further constraints are imposed through hardware limitations (finite signal bandwidth, noise in the analogue frontend). Combined effects of both error sources lead in practice to non-Gaussian distributions of introduced measurement inaccuracies.

Another important aspect which makes particle filtering a suitable method for position estimation is the fact that often multiple information sources (sensor-fusion) can be combined in order to improve the accuracy. Particularly in tracking applications, system states that go beyond the position (like velocity or orientation) are of interest. Bayesian estimators provide the mechanisms for mathematical description of such state estimation problems. For many problems no analytical solution of the Bayes estimator can be found. The formalism which numerically approximates the Bayesian estimator is called *particle filtering*.

The rest of this report is structured as follows. In Section II the mathematical formulation of the state estimation problem is reviewed. Fundamentals of Bayesian estimation are given in Section III. Particle filtering (PF) as a method for actual calculation the Bayesian estimator is described in Section IV. Finally two examples of PF in positioning and tracking applications is provided in Section V.

II. System Model

Tracking of system states $\boldsymbol{\theta}$ implies the notion of time-variability. In the following we will assume a time-variant time-discrete model. Two equations are required to describe it. The dynamical model (Eq.1) describes the (generally nonlinear) mapping of two consecutive system states $\boldsymbol{\theta}_k \rightarrow \boldsymbol{\theta}_{k+1}$ of the same dimension $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$. $\mathbf{w}_k$ is the noise term providing for uncertainties in the dynamical model.

$$\boldsymbol{\theta}_{k+1} = f(\boldsymbol{\theta}_k) + \mathbf{w}_k \tag{1}$$

The measurement model (Eq.2) defines how internal system states map to the observables $\mathbf{z}$. Also h is a nonlinear function. The dimension of $\mathbf{z}$ can be different from the dimension of $\boldsymbol{\theta}$. Moreover the dimension of $\mathbf{z}$ can be time varying $h_k : \mathbb{R}^d \rightarrow \mathbb{R}^{m_k}$. This is particularly important for sensor-fusion applications where not all sensors can provide inputs all the time. An example for this would be the combination of external inputs (e.g. GPS) and some local positioning system. Under certain circumstances GPS input will not be available (e.g. indoor operation) so that the sate estimation relies only on the available obervables. The noise vector $\mathbf{v}$ represents uncertainties introduced in the measurement process.

$$\mathbf{z}_k = h_k(\boldsymbol{\theta}_k) + \mathbf{v}_k \tag{2}$$

Let a time discrete data-record of observations be $\mathbf{z}_{0:k} = \{\mathbf{z}_0, \ldots, \mathbf{z}_k\}$. The goal of the estimation procedure is the inference of the unknown system states from observed data $\mathbf{z}_{0:k} \rightarrow \hat{\boldsymbol{\theta}}(k; \mathbf{z}_{0:k})$. Applications like object tracking require an online estimation of the current system state. For this reason a definition of recursive procedure is desired, which derives the new state estimate from the previous estimate and the current observation (Eq.3). $\mathbf{S}$ in Eq. 3 denotes an auxiliary quantity which might be necessary for the recursive calculation. Like in all recursive procedures a reasonable initial estimate $\boldsymbol{\theta}_0$ is required.

$$\hat{\boldsymbol{\theta}}_k = F(\hat{\boldsymbol{\theta}}_{k-1}, \mathbf{z}_k, \mathbf{S}_{k-1}) \tag{3}$$

Having presented the problem, the question arises which classes of algorithms are well suited for its solution. An extensive overview of different solution approaches is given in [2]. The options range from modification of off-line algorithms (recursive least squares (RLS) to model reference techniques (adaptive filtering) and gradient-based methods. Kalman Filter, Extended Kalman Filter and many modified versions of them are popular approaches, yield however suboptimal solutions for a general class of nonlinear/non-Gaussian problems. A popular method to cope with this is application of Bayesian inference methods.

III. Bayesian Estimation

Coventionally the unknown state space vector $\boldsymbol{\theta}$ is assumed to be unknown, but deterministic quantity. The Bayesian approach departs from such an assumption and assumes it to be a random variable (RV). Consequently the parameter to be estimated is described using a multinomial probability density function (pdf) $p(\boldsymbol{\theta})$.

In this way a probabilistic framework for parameter estimation is established. It is characterized by two properties which are not available compared to deterministic approach. Firstly, by describing $\boldsymbol{\theta}$ as a RV it is implied that the estimated parameter(s) include some uncertainties. Secondly, as it will be shown later, the formalism allows for the introduction of prior knowledge through appropriate selection of the initial parameter distribution $p_0(\boldsymbol{\theta})$.

The formalism of the Bayesian estimator is based on the basis statistical concept of conditional probabilities. Therefore a review of the Bayes Rule is appropriate before continuing the discussion on the estimator itself.

A. Bayes rule

Firstly published by Thomas Bayes in *Essay Towards Solving a Problem in the Doctrine of Chances* (1764) the Bayes rule introduced the concept of the mutual dependence of conditional probabilities of two events.

Assume a set of n disjoint discrete events $\{E_i\}_{i=1}^n$. Further a probability $P(E_i)_{i=1}^n$ is assigned to each of these events. If the events are *not* independent (e.g. drawing of numbered balls from a pot), then a specific random realisation (lets say of event E_j) alters the probabilities of all other subsequent events. This notion is formalized in the concept of conditional probabilities. If event E_j has occured, then the probability of the event E_k is denoted as the probability of E_k given E_j. Mathematically this is noted as $P(E_k|E_j)$. A *joint* probability that both events occure, denoted $P(E_j, E_k)$ can be calculated as $P(E_j, E_k) = P(E_j|E_k)P(E_k)$ or alternatively as $P(E_j, E_k) = P(E_k|E_j)P(E_j)$. Given that the left hand side of the last two expressions is equal we can rewrite this as Eq. 4

$$\begin{aligned} P(E_j|E_k) &= \frac{P(E_k|E_j)P(E_j)}{P(E_k)} \\ P(E_k|E_j) &= \frac{P(E_j|E_k)P(E_k)}{P(E_j)} \end{aligned} \tag{4}$$

The term in the denominator of 4 is a scaling term and can be calculated as $P(E_j) = \sum P(E_j|E_i)P(E_i)$.

The same considerations also apply to continuous case where discrete events are replaced by pdfs. Given two RV X and Y the Bayes rule applies as in Eq. 5. Instead of the sum an integral is calculated in the normalizing factor.

$$p(y|x) = \frac{p(x|y)p(y)}{p(x)} = \frac{p(x|y)p(y)}{\int p(x|y)p(y)dy} \tag{5}$$

Equation 5 already carries the notion of the most important concepts for the Bayesian estimator.

The second term in the nominator $p(y)$ is called the *prior* distribution. As the name suggests it models some a priori knowledge or assumption about the distribution of the parameter. The first term of the denominator is called *likelihood*. The updated pdf $p(y|x)$ is called *posterior*.

B. Bayesian estimator

The essence of the Bayesian estimator is given already in Eq.5. Formulating it for for online estimation of a time varying system (Sec.II) results in a recursive procedure which requires two steps.

The first step, prediction, provides for a prior estimate of the parameter distribution. Before the first iteration, the prior is not calculated but provided as an initializing pdf $p_0(\boldsymbol{\theta})$. After that the prior (also called predictive pdf) is calculated by employing the assumed dynamical model of the system (1). Using again the notation $\boldsymbol{\theta}_k$ for unknown system states at the time instant k and the corresponding observations $\mathbf{z}_k$ the calculation of the prior distribution is given in Eq.6

$$p(\boldsymbol{\theta}_{k+1}|\mathbf{z}_{0:k}) = \int p(\boldsymbol{\theta}_{k+1}|\boldsymbol{\theta}_k)p(\boldsymbol{\theta}_k|\mathbf{z}_{0:k})d\boldsymbol{\theta}_k \quad (6)$$

The next step, called update, calculates the posterior pdf using the above described Bayes rule. For completeness the equation is given (again) in 7.

$$p(\boldsymbol{\theta}_k|\mathbf{z}_{0:k}) = \frac{p(\mathbf{z}_k|\boldsymbol{\theta}_k)p(\boldsymbol{\theta}_k|\mathbf{z}_{0:k-1})}{p(\mathbf{z}_k|\mathbf{z}_{0:k-1})} \quad (7)$$

The question remains how a parameter estimate $\hat{\boldsymbol{\theta}}$ is derived from the posterior density. One possibility includes the calculation of the average of the calculated distribution (Eq.8). This is the maximum likelihood output of the estimator. It is also possible to provide a maximum-a-posteriori (MAP) value of $\hat{\boldsymbol{\theta}}$, which is essentially the peak of the calculated distribution (Eq.9).

$$\hat{\boldsymbol{\theta}}_{\boldsymbol{k},\boldsymbol{ML}} = E(p(\boldsymbol{\theta}_k|\mathbf{z}_{0:k})) \quad (8)$$

$$\hat{\boldsymbol{\theta}}_{\boldsymbol{k},\boldsymbol{MAP}} = argmax(p(\boldsymbol{\theta}_k|\mathbf{z}_{0:k})) \quad (9)$$

For the case of symmetrical distributions $\hat{\boldsymbol{\theta}}_{\boldsymbol{k},\boldsymbol{ML}} = \hat{\boldsymbol{\theta}}_{\boldsymbol{k},\boldsymbol{MAP}}$. The variance of the posterior distribution is also of interest, as it suggests the confidence level of the current estimate.

Figure 1 shows a scheme of the iterative concept of the Bayesian estimator. The concept is simple but has a significant drawback. Analytical solutions of 6 and 7 are known for few special cases. Most notably this is the linear system model combined with Gaussian distributions, for which the Kalman Filter is the exact solution. In all other cases, numerical approximations need to be calculated. The tool which provides this possibility is the particle filter.

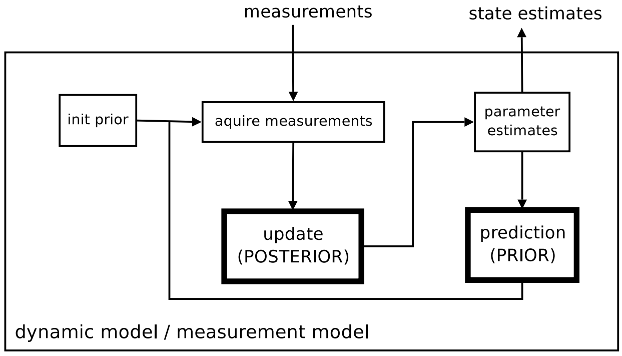

Fig. 1. Flow diagram of the iterative Bayes estimator

IV. Particle Filtering

A. Approximating distribution functions

In order to obtain a numerical representation of a probability density functions of interest, the concept of weighted particles is introduced. Particles $\mathbf{x}_i$ in this context are samples from the state space. A weight is assigned to every particle such that a set of M pairs $\{w_i, \mathbf{x}_i\}$ approximates a desired posterior pdf as given in Eq. 10, where δ denotes the Dirac impulse. The associated weights need to be normalized such that they fulfil $\sum_{i=1}^{M} w_i = 1$.

$$p(\boldsymbol{\theta}_k|\mathbf{z}_{0:k}) \approx \sum_{i=1}^{M} w_i^{(k)}\delta(\mathbf{x} - \mathbf{x}_i^{(k)}) \quad (10)$$

B. Sequential importance sampling (SIS)

The task of discrete state estimation in Bayesian context is to recursively update the estimated posterior pdf. This is also what is calculated in the particle filter namely a recursive transition from $\{w_i^{k-1}, \mathbf{x}_i^{k-1}\}$ at time instant $k-1$ to $\{w_i^k, \mathbf{x}_i^k\}$ at the next time instant k, under consideration of newly obtained measurements (Eq.11). The algorithm for the operator $\mathcal{S}$ is called sequential importance sampling (SIS).

$$\{w_i^k, \mathbf{x}_i^k\} = \mathcal{S}(\{w_i^{k-1}, \mathbf{x}_i^{k-1}\}, \mathbf{z}_k) \quad (11)$$

The function from which the particle values are drawn is called *importance* function and will be denoted as $\pi(\boldsymbol{\theta})$. If any apriori knowledge about the distribution of system state is available or can be reasonably assumed, the importance function is selected such to reassemble this knowledge. If no prior information is available $\pi(\boldsymbol{\theta})$ is selected as a uniform distribution in the supported space of the system. At each iteration a set of M samples are drawn from $\pi(\boldsymbol{\theta})$[1]. This step is followed by a weight update. A more detailed discussion on the selection of the importance function can be found in [4].

The concept of importance sampling implies that the importance function at time k can be seen as the previous

[1]This makes the particle filter a sequential Monte Carlo method (SMC) a name often used to refer to the PF itself.

importance function at $k-1$ augmented by the new state derived from the new set of observations (Eq.12).

$$\pi(\boldsymbol{\theta}_{0:k}|\mathbf{z}_{0:k}) = \pi(\boldsymbol{\theta}_k|\boldsymbol{\theta}_{0:k-1}, \mathbf{z}_{0:k})\pi(\boldsymbol{\theta}_{0:k-1}|\mathbf{z}_{0:k-1}) \tag{12}$$

If this assumption holds, it can be derived [4] that the weights at each iteration are updated as given in Eq. 13.

$$w_k = w_{k-1}\frac{p(\mathbf{z}_k|\boldsymbol{\theta}_k)p(\boldsymbol{\theta}_k|\boldsymbol{\theta}_{k-1})}{\pi(\boldsymbol{\theta}_k|\boldsymbol{\theta}_{0:k-1}, \mathbf{z}_k)} \tag{13}$$

C. Degeneracy and resampling

The SIS algorithm as described above tends to assign negligible weights to most of the particles after just a couple of iterations. Thus a significant computational effort is spent on recalculating weights for particles, which effectively contribute little to the knowledge of the posterior density. This is refered to as a degeneracy problem. A brute force approach to deal with this would be drawing an extremely large number of particles from $\pi(\boldsymbol{\theta})$.

A better approach is to employ *resampling*. During resampling particles with small weights are discarded and particles with larger weights reproduced. All new particles are assigned the same equal weight of $w_i = \frac{1}{M}$. It is not necessary to resample the particles after every iteration. An approximate measure [4] indicating degeneracy is called the *effective particle size* (Eq. 14). As small N_{eff} indicates degeneracy of the particles the resampling procedure is invoked when this value falls below a user set threshold $N_{eff} < N_{th}$.

$$N_{eff} = \frac{1}{\sum_{i=1}^{M} w_i^2} \tag{14}$$

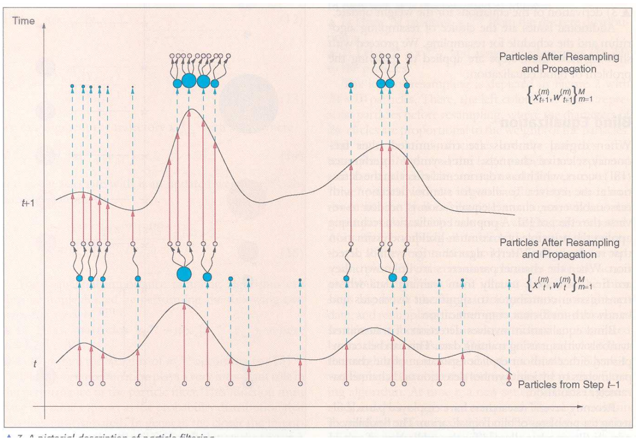

Fig. 2. Three steps of the particle filtering [3]

Figure 2 shows two iterations of a SIS algorithm with resampling. Note that the ordinate represents time and the abscissa represents the one-dimensional system state. The size of a circle (particle) indicates the corresponding weight. Beginning with a set of particles with equal weights (lowest set of circles), the weights are updated by calculating 13 (lower blue circles). The particles are then resampled such that particles with low weights disappear from the plot and strongly weighted particles are reproduced proportionally to their weight. All resampled particles are assigned the same weight an the procedure is iterated.

Several modified versions of the basic particle filter presented here exist. A review of these techniques is summarized in [4]. An important class is the Rao-Blackwellized Filter [5]. Given that a subset of states is linear in the system model, the computational complexity can be reduced by separately calculating the linear and nonlinear states.

V. Selected Examples

This section provides an overview of selected works on Bayesian methods in positioning. The main idea is for the reader to obtain a feeling about tackling different positioning problems using the same methodology. The examples are selected such to provide an insight in a diversity of possible applications. The reader is encouraged to refer to the cited references for a detailed analysis.

A. UWB indoor positioning

Due to inherent good time resolution of wideband signals they are well suited for positioning applications. A synchronized ultra-wideband (UWB) system provides for an accurate time of arrival (TOA) measurements thus making an accurate target location possible. However a number of issues makes positioning in indoor scenarios a difficult task. Obstruction and attenuation of the leading edge due to no-line-of-sight (NLOS) propagation is one of the main factors.

In [1] a Bayesian framework is derived to cope with the effects of biased TOA estimates. Firstly the fact is utilized that bias, unlike other error sources, is spatially correlated among base units and depends on the propagation environment and the mobility pattern. Due to unpredictability of the propagation environment also sudden changes in the detected bias can be expected. To cope with this situation a prior is selected as a mixture of a Gaussian and exponential distributions. Then the problem is formulated in the state-space such that biases for all ranging links are explicitly formulated as state variables to be estimated. In order to relax the constraints on the pulse detectability the strongest path in each ranging link is measured. With correct bias estimation, the peak, which is more stable then attenuated first path, provides a valuable distance reference.

Note that this approach does not work for the static case, as the modeled geometrical evolution of biases is a key value to be estimated. A significant simplification is introduced with the assumption of linear movement model. This is a simple approach to a difficult problem of predicting pedestrian indoor movements. A modified version of the particle filter, called modified regularized particle filter (MRPF) ([4], [1]) is employed for tracking the posterior densities. The authors report promising simulation results for realistic indoor channel assumptions, pedestrian moving speeds and low position update rates.

B. Map-aided navigation

The most important aspect in this application is the possibility of the particle filter to combine substantially different input sources. In [5] a procedure for determining the position of an underwater vehicle is described. Data from an inertial navigation system (INS), composed of velocity and acceleration vectors are combined with a measured depth at the given position. As underwater terrain data-base is available it is possible to combine the two information sources to derive a system model, which has the unknown position coordinates as the state variables.

A similar principle applies also in [6] where GPS inputs are combined with radar distance measurements and an accurate sea chart. Further applications of particle filters in positioning include systems with bearing only information [7].

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