Bayesian Methods in Positioning Applications

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Outline

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Problem Statement (1/2)

- System description in two steps
  - Find system model (modeling)
  - Find the particular realization of the model (identification; parameter estimation $\hat{\theta}$)
- Time variant system parameters require recursive estimation
  - on-line vs. off-line data processing

Problem Statement (2/2)

- Basic notation
  - Time-discrete data record $\mathbf{z}_{0:k} = \{\mathbf{z}_0, \ldots, \mathbf{z}_k\}$
  - With N-dimensional observation $\mathbf{z}_k = \{z_k^1, \ldots, z_k^N\}$
  - Inference of the model parameter $\mathbf{z}_{0:k} \longrightarrow \hat{\boldsymbol{\theta}}(k; \mathbf{z}_{0:k})$
  - Recursive estimation $\hat{\boldsymbol{\theta}}_k = F(\hat{\boldsymbol{\theta}}_{k-1}, \mathbf{z}_k, \mathbf{S}_{k-1})$
- Initial condition(s) - selection of $\boldsymbol{\theta}_0$

Solution Alternatives

- Modification of off-line estimation methods
  - RLS
- Stochastic approximation
  - Gradient-based methods
- Model reference techniques and pseudolinear regression
  - Adaptive adjustment through comparison of two systems
  - Extension of linear regression (LS) methods
- Bayesian inference
  - Estimation of distribution of θ

Introduction to Bayesian Estimation

- Bayesian approach
  - Probabilistic framework for recursive state estimation
  - In Bayesian estimation $\boldsymbol{\theta}$ is treated as a random variable with $p(\boldsymbol{\theta})$
  - By defining $p(\boldsymbol{\theta})$ estimationuncertaintiess are implied
  - The formalism allows introduction of *prior* knowledge of $p(\boldsymbol{\theta})$
  - Relying on a Bayes rule updated $p(\boldsymbol{\theta})$ isinferredd from prior (predicted) $p(\boldsymbol{\theta})$ combined with incoming observations

Bayes Rule and Contemporary Science

- Published in T. Bayes *Essay Towards Solving a Problem in the Doctrine of Chances* (1764)

- Steam engine of J. Watt (1769)
- B. Franklin defines the concept of positive and negative polarity (1752)
- L. Galvani and frogs (1770)
- K.F. Gauss was born (1777)

Bayes Rule

- Assume a disjoint set of n discrete events $\{E_i\}_{i=1}^{n}$
- Probabilities $P(E_i)_{i=1}^{n}$ are assigned to each event
- If events are *not independent* observing that event E_k hasoccurredd, probabilities of all other events $\{E_i\}_{i=1}^{n}$ will change
- This is described using *conditional probability* $P(E_i|E_k)$
- $P(E_i, E_k) = P(E_i|E_k)P(E_k) = P(E_k|E_i)P(E_i)$
- $P(E_i|E_k) = \frac{P(E_k|E_i)P(E_i)}{P(E_k)}$

Bayesian Estimator (1/4)

Two equations describe the discrete-time, time-variant, d-dimensional system

- *Motion model*
  $$\boldsymbol{\theta}_{k+1} = f(\boldsymbol{\theta}_k) + \mathbf{w}_k$$
  $$\boldsymbol{\theta}_k, \mathbf{w}_k \in \mathbb{R}^d, f : \mathbb{R}^d \longrightarrow \mathbb{R}^d$$
- *Measurement model*
  $$\mathbf{z}_k = h_k(\boldsymbol{\theta}_k) + \mathbf{v}_k$$
  $$\mathbf{z}_k, \mathbf{v}_k \in \mathbb{R}^{m_k}, h_k : \mathbb{R}^d \longrightarrow \mathbb{R}^{m_k}$$

Bayesian Estimator (2/4)

Notation

- Conditional pdf
  $p_{k|l}(\boldsymbol{\theta}_k) = p(\boldsymbol{\theta}_k|\mathbf{z}_1, \ldots, \mathbf{z}_l)$
- Likelihood function
  $L_k(\boldsymbol{\theta}_k) = p(\mathbf{z}_k|\boldsymbol{\theta}_k) = p_{v_k}(\mathbf{z}_k - h_k(\boldsymbol{\theta}_k))$
- Transition pdf
  $\phi_k(\boldsymbol{\theta}_{k+1}|\boldsymbol{\theta}_k) = p_{w_k}(\boldsymbol{\theta}_{k+1} - f(\boldsymbol{\theta}_k))$

Bayesian Estimator (3/4)

- *Step 0*
  Select **initial distribution** $p_{0|0}$ and set $k = 1$
- *Step 1*
  Compute the **predictive pdf**
  $p_{k|k-1}(\boldsymbol{\theta}_k) = \int_{\mathbb{R}^d} p(\boldsymbol{\theta}_k|\boldsymbol{\theta}_{k-1})p(\boldsymbol{\theta}_{k-1}|\mathbf{z}_1, \ldots, \mathbf{z}_{k-1})d\boldsymbol{\theta}$
- *Step 2*
  Compute the **posterior pdf**
  $p_{k|k}(\boldsymbol{\theta}_k) = \frac{p(\mathbf{z}_k|\boldsymbol{\theta}_k)p(\boldsymbol{\theta}_k|\mathbf{z}_1,\ldots,\mathbf{z}_{k-1})}{p(\mathbf{z}_k|\mathbf{z}_1,\ldots,\mathbf{z}_{k-1})} \propto p_{k|k-1}(\boldsymbol{\theta}_k)L_k(\boldsymbol{\theta})$
- *Step 3*
  Output $\hat{\boldsymbol{\theta}}_{\boldsymbol{k}}$ and variance $V(p_{k|k}(\boldsymbol{\theta}_k))$
- *Step 4*
  Increment k and repeat from Step 1

Bayesian Estimator (4/4)

- Calculation of $\hat{\boldsymbol{\theta}}_{\boldsymbol{k}}$
  $\hat{\boldsymbol{\theta}}_{\boldsymbol{k}} = E(p_{k|k}(\boldsymbol{\theta}_k))$
  $\hat{\boldsymbol{\theta}}_{\boldsymbol{k}} = argmax(p_{k|k}(\boldsymbol{\theta}_k))$ (MAP)
- Problem: it is difficult to find an analytical solution to this problem!
- If the noise pdfs are independent, white and zero-mean Gaussian and the measurement equation linear in $\boldsymbol{\theta}$, it can be derived that the Bayesian approach reduces to the Kalman Filter
- A general algorithm which approximates the pdfs with arbitrary accuracy is the particle filter

Particle Filter (1/4)

- Two steps require intractable integration
  - Posterior pdf computation
  - Expectation of the posterior pdf
- Numerical means for the tracking the evolution of the pdfs
- The pdf is approximated as weighted sum of samples in the space state (particles)
  $p(\boldsymbol{\theta}) \approx \sum_{m=1}^{M} w^{(m)}\delta(x - x^{(m)})$
- Recursive update
  - Prediction: Calculate the predictive pdf using the motion model
  - Update: Weights update with normalization

Particle Filter (2/4)

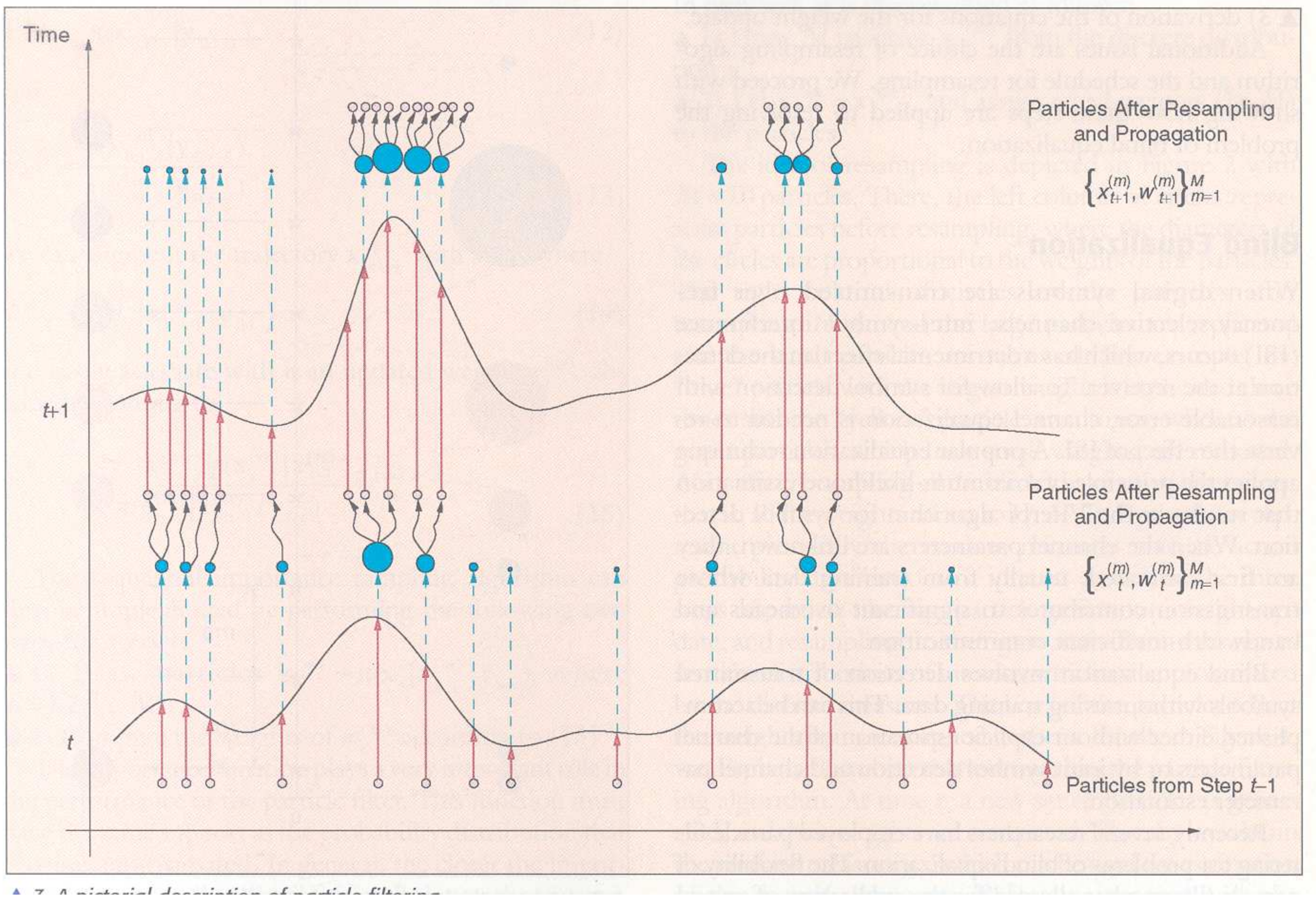
- *Initialization*
  Select an importance function π and create a set of particles $\mathbf{x}_0^{(m)} \sim \pi$
- *Measurement update*
  Weight update according to the likelihood
  $w_k^{(m)} = w_{k-1}^{(m)} \pi(\boldsymbol{\theta}|\mathbf{z})$
  Normalize the weights $w_k^{(m)} := w_k^{(m)} / \sum_m w_k^{(m)}$
  Calculate the estimate $\hat{\boldsymbol{\theta}}_k \approx \sum_{m=1}^{M} w_k^{(m)} \boldsymbol{\theta}_k^{(m)}$
- *Prediction*
  $\boldsymbol{\theta}_{k+1} = h(\boldsymbol{\theta}_k) + v_k$

Particle Filter (3/4)

- Measurement update
- Resampling
- Prediction

- Weights tend to concentrate in few particles
- Draw P particles from the *current* particle with probabilities proportional to the corresponding weight; set weights for the new particle as $w_k = \frac{1}{P}$
- Optionally resampling only if a given criterion is fulfilled $N_{eff} = \frac{1}{\sum_i (w_k^{(i)})^2} < N_{th}$

Particle Filter (4/4)



MATLAB example

- System equations

$$\theta_k = 0.5\theta_{k-1} + 25\frac{\theta_{k-1}}{1+\theta_{k-1}^2} + 8cos(1.2k) + w_k$$

$$z_k = \frac{\theta_k^2}{20} + v_k$$

- Simulation parameters

$$\sigma_w^2 = 10$$
$$\sigma_v^2 = 1$$
$$\sigma_\pi^2 = 5$$
$$M = 500$$

Pro and Contra

- This algorithm is known under many names!
- Pro:
  Non-linear sate model needs no approximations
  Non-Gaussian distributions θ
  Combines different observations in a single model
  Variable dimension of the observation vector no problem
- Contra:
  Computationally expensive
  Selection of prior distribution

Bayesian Estimation in Positioning Applications (1/2)

- Application: Positioning, Navigation, Tracking
- Type of object: Persons, Robots, Vehicles
- Supplementary information: Road map, terrain profile

- State space:
  $\boldsymbol{\theta} = [\mathbf{p}, \phi, \mathbf{v}, \dot{\mathbf{v}}, \ldots]^T$
- Observations:
  $\mathbf{z} = [\mathbf{p}_{ext}, \phi_{ins}, \dot{\mathbf{v}}_{ins}, \gamma_{image}, \gamma_{radar}, \mathbf{S}_{ss}, \ldots]^T$

Bayesian Estimation in Positioning Applications (2/2)

- Current research directions
  - Motion models
    Prediction of pedestrian movements
  - Complexity reduction
    Employ KF for linear states
  - Parallelization
    Distributed PF in a sensor network

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