Fundamentals of Estimation Theory

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Outline

- What is Estimation?
- Applications
- Mathematical Formulation
- Minimum Variance Unbiased Estimator
- Cramer Rao Bound

Estimation

It is a rule / function that assigns a value θ to for each realization of x. The estimate of θ is the value obtained for a given realization x.

Applications

- Radar
- Sonar
- Speech
- Image analysis
- Biomedicine
- Communications
- Control
- Seismology

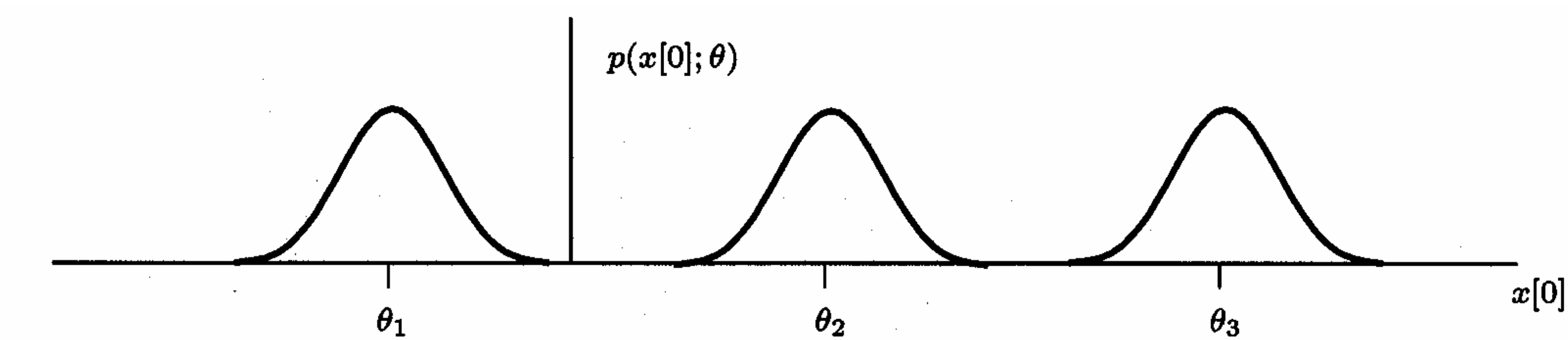
Mathematical Formulation

- To model random data mathematically, we need PDF
- A PDF is parameterized by unknown parameter θ

$$p(x[0];\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} exp[-\frac{1}{2\sigma^2}(x[0]-\theta)^2]$$

Semicolon in the above Eq. Denotes the family of PDFs depending on parameter θ

Mathematical Formulation



Unbiased Estimator

Estimator which on average yield the true value of parameter

Mathematically: $E(\hat{\theta}) = \theta$

Unbiased Estimator: $\hat{A} = \frac{1}{N} \sum^{N-1} x[n]$

Expected Value: $E(\hat{A}) = A$

Unbiased estimator does not necessarily mean a good estimator but on average it will converge to true value.

Biased Estimator

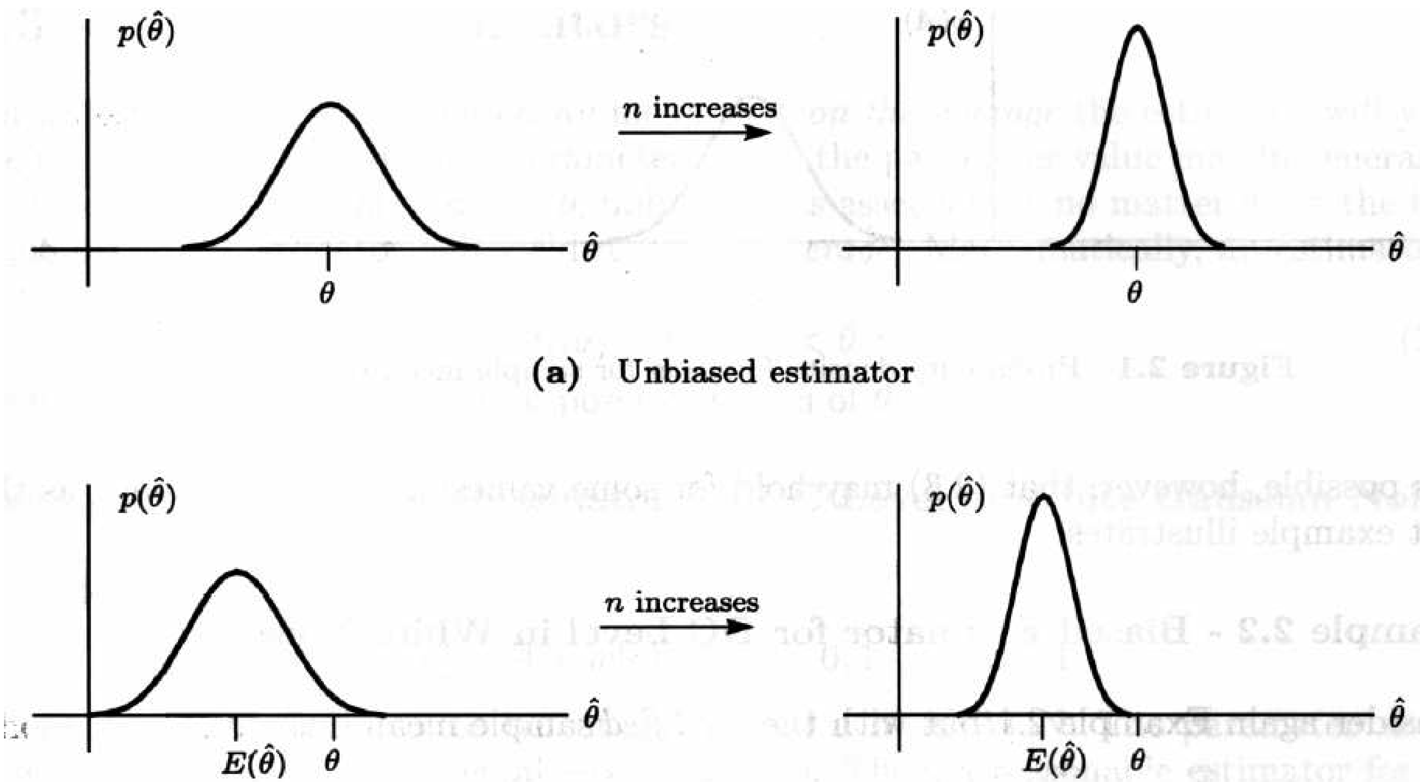
Biased Estimator: $$\check{A} = \frac{1}{2N}\sum_{n=0}^{N-1} x[n]$$

Expected value: $$E(\check{A}) = \frac{1}{2}A$$

Bias is defined as:

$$b(\theta) = E(\hat{\theta}) - \theta$$

Effect of Biasing



(a) Unbiased estimator

(b) Biased estimator

Minimum Variance Unbiased Estimator

<u>Mean Square Error</u>

It is the natural choice but leads to biased estimator.

Mathematically : $$mse(\hat{\theta}) = var(\hat{\theta}) + b^2(\theta)$$

<u>Minimum Variance</u>

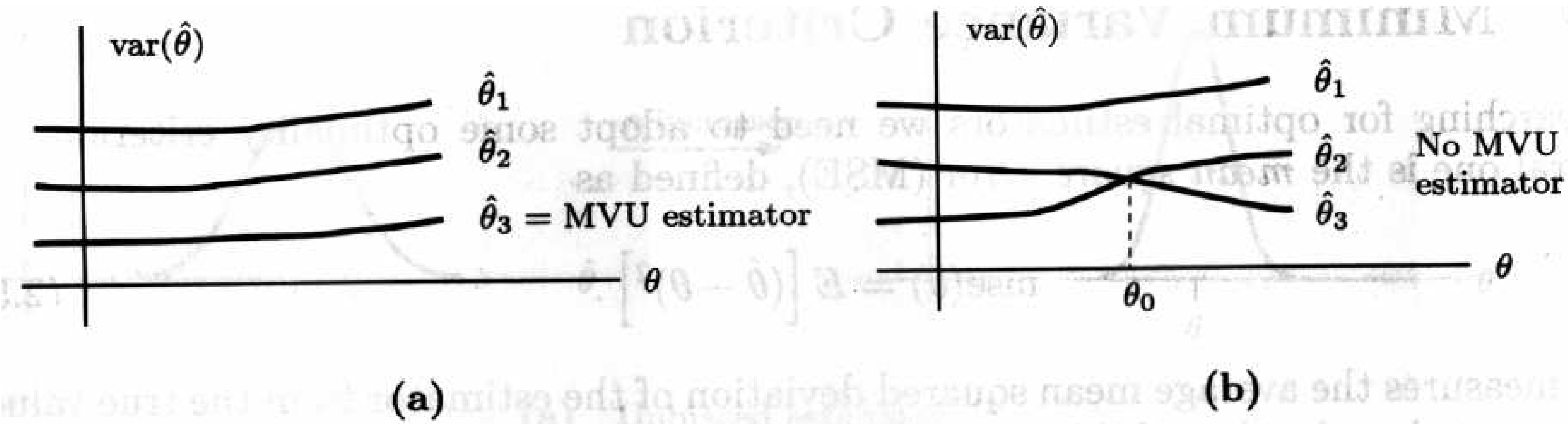
It is obtained by forcing bias term to zero in mse estimator.

Minimum Variance Unbiased Estimator

- Existence of MVUE

  Exits only if class of PDF does not change with Parameters
- Finding MVUE
  1. By CRLB
  2. By RBLS theorem

Existence of Minimum Variance Unbiased Estimator



Likelihood Function

- When PDF is viewed as function of unknown parameter
- Its sharpness determines the accuracy of estimator

Loglikelihood Function

- It is used to determine the sharpness
- Negative of the second derivative is used

Fisher Information

Expected value of the second derivative of Loglikelihood Function.

$$I(\theta) = -E\left[\frac{\partial^2 \ln p(x[0];\theta)}{\partial \theta^2}\right]$$

- Always Nonnegative
- Additive for independent observations

Asymptotic Fisher Information

For the cases where it is difficult to evaluate the CRLB then asymptotically Fisher Information can be expressed as:

$$[\mathbf{I}(\theta)]_{ij} = \frac{N}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial \ln P_{xx}(f;\theta)}{\partial \theta_i} \frac{\partial \ln P_{xx}(f;\theta)}{\partial \theta_j} df$$

$P_{xx}(f;\theta)$ is the PSD of the process with explicit depdence on θ . Also $x[n]$ is zero mean.

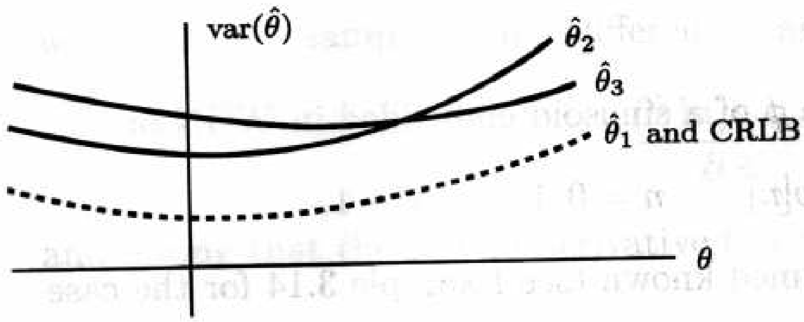
Cramer Rao Bound

If a process has regular PDF then variance of any unbiased estimator $\hat{\theta}$ must satisfy

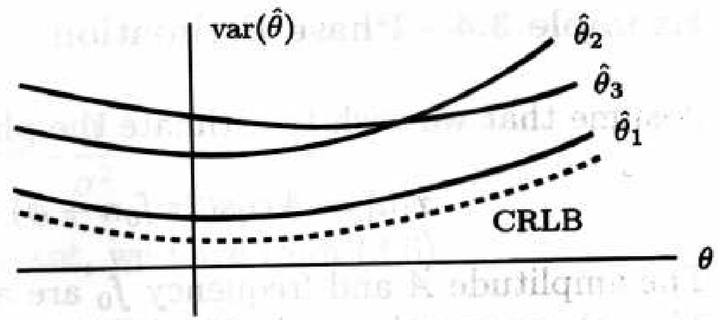
$$var(\hat{\theta}) \geq \frac{1}{-E[\frac{\partial^2 \ln p(x[0];\theta)}{\partial \theta^2}]}$$

Denominator of the above Eq. is Fisher Information.
CRLB attained when variance is reciprocal of Fisher Information

Cramer Rao Bound



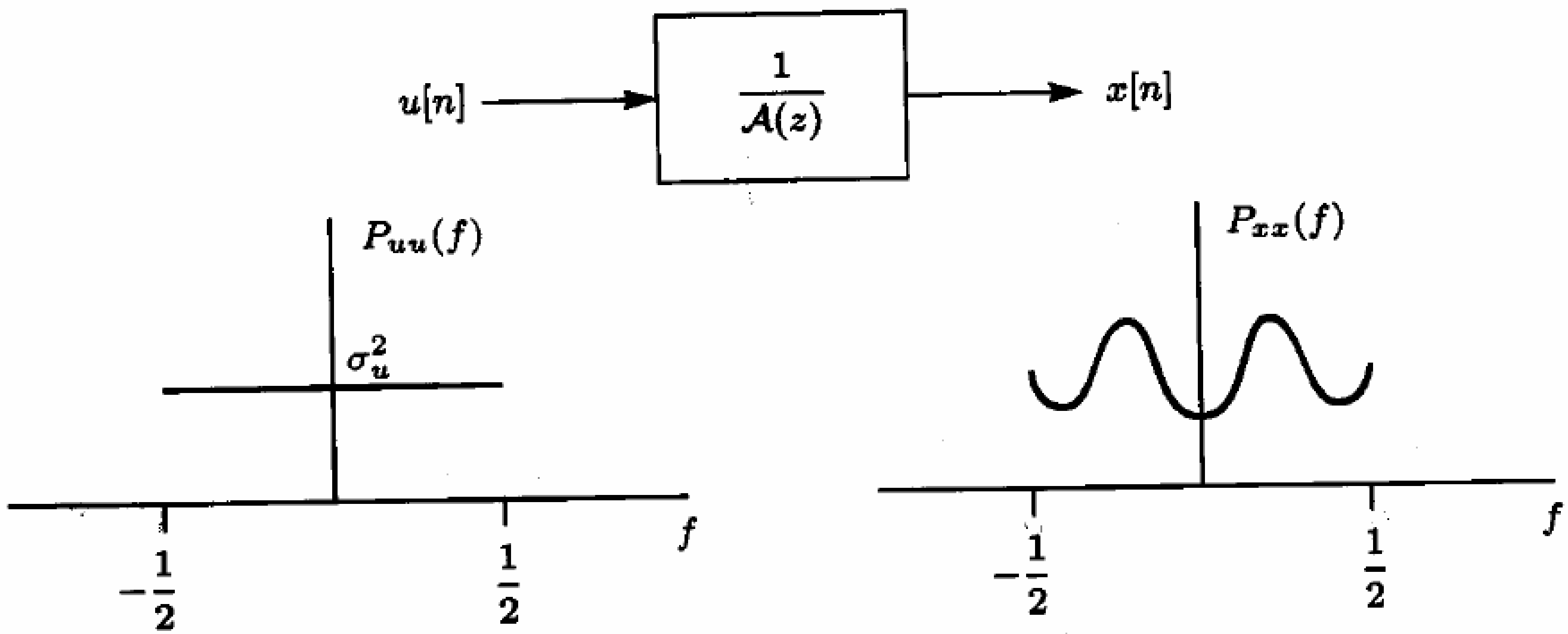
(a) $\hat{\theta}_1$ efficient and MVU



(b) $\hat{\theta}_1$ MVU but not efficient

Example AR Gaussian Parameters

Autoregressive model



Example AR Gaussian Parameters

- Estimated Power Spectral Desity (PSD)

$$\hat{P}_{xx}(f) = \frac{\hat{\sigma}_u^2}{|1 + \sum_{m=1}^{p} \hat{a}[m]exp(-j2\pi fm)|^2}$$

It is difficult to calculate CLRB for this case but asymptotic CRLB can be found.

Example AR Gaussian Parameters

- Fisher Information Matrix is:

$$I(\theta) = \left[\begin{array}{cc} \frac{N}{\sigma_u^2}\boldsymbol{R}_{xx} & \boldsymbol{0} \\ 0^T & \frac{N}{2\sigma_u^4} \end{array} \right]$$

Where $[\mathbf{R}_{xx}^{-1}]_{kk}$ is the *pxp* Toeplitz autocorrelation matrix and ***0*** is the *px1* vectors of zeros

Example AR Gaussian Parameters

CRLB for Parameters:

$$var(\hat{a}[k]) \geq \frac{\sigma_u^2}{N}[\mathbf{R}_{xx}^{-1}]_{kk}$$

CRLB for Variance:

$$var(\hat{\sigma_u^2}) \geq \frac{2\sigma_u^4}{N}$$