Introduction to Higher-Order Spectra Analysis

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Outline

- Motivation
- Moments and Cummulants (basic definitions, properties,..)
- Moments and Cummulants of Stationary Processes
- Cummulant Spectra (power spectrum,..)
- Linear Phase Shifts
- Regression Coefficient (linear system identification)
- Nonminimum Phase LTI Systems
- Coherency Function
- Nonlinear System Identification

Motivation

- Supress Gaussian noise processes of unknown spectral characteristics in detection and parameter estimation problems

- Reconstruction of magnitude and phase information of signals and systems

- Detect and characterize nonlinearities in time series

Moments and Cummulants

$$Mom[x_1^{k_1}, x_2^{k_2}, \dots, x_n^{k_n}] = E\left\{x_1^{k_1} x_2^{k_2} \dots x_n^{k_n}\right\} =$$

$$= (-j)^r \left. \frac{\partial^r \phi(\omega_1, \omega_2, \dots \omega_n)}{\partial \omega_1^{k_1} \partial \omega_2^{k_2} \dots \partial \omega_n^{k_n}} \right|_{\omega_1 = \omega_2 = \cdots = \omega_n = 0}$$

joint charact. Function (Fourier transform of the joint pdf!)

$$\phi(\omega_1, \dots \omega_n) = E\left\{\exp\left(j\left(\omega_1 x_1 + \cdots + \omega_n x_n\right)\right)\right\} =$$

$$= \int \cdots \int \exp\left(j\left(\omega_1 x_1 + \cdots + \omega_n x_n\right)\right) p(x_1, \dots, x_n) dx_1 \dots dx_n$$

joint pdf

Moments and Cummulants

$$Cum[x_1^{k_1}, x_2^{k_2}, \ldots, x_n^{k_n}] =$$

$$= (-j)^r \left. \frac{\partial^r \psi(\omega_1, \omega_2, \ldots \omega_n)}{\partial\omega_1^{k_1} \partial\omega_2^{k_2} \ldots \partial\omega_n^{k_n}} \right|_{\omega_1 = \omega_2 = \cdots = \omega_n = 0}$$

Natural log. of the joint charact. Function

$$\psi(\omega_1, \ldots, \omega_n) = \ln[\phi(\omega_1, \ldots \omega_n)]$$

Moments and Cummulants

$$m_1 = Mom[x] = E\{x\}$$

$$m_2 = Mom[x,x] = E\left\{x^2\right\}$$

.

.

$$\boxed{c_1} = Cum[x] = -j\frac{d\left(\ln\phi(\omega)\right)}{d\omega}\bigg|_{\omega=0} = -j\frac{1}{\phi(\omega)}\frac{d\phi(\omega)}{d\omega}\bigg|_{\omega=0} = \boxed{m_1}$$

$$\boxed{c_2} = Cum[x,x] = (-j)^2\frac{d^2\psi(\omega)}{d\omega^2}\bigg|_{\omega=0} =$$

$$= -\frac{d}{d\omega}\left(\frac{1}{\phi(\omega)}\frac{d\phi(\omega)}{d\omega}\right)\bigg|_{\omega=0} = \boxed{m_2 - m_1^2}$$

Moments and Cummulants

Leonov-Shiryaev formula

$$Cum\left[x_1,\ldots,x_n\right]=\sum(-1)^{p-1}(p-1)!\cdot$$

$$\cdot E\left\{\prod_{i\in s_1}x_i\right\}E\left\{\prod_{i\in s_2}x_i\right\}\cdots E\left\{\prod_{i\in s_p}x_i\right\}$$

$$\left(s_1,s_2,\ldots,s_p\right),\ p=1,2,\ldots,n$$

Moments and Cummulants

$$Cum\left[x_1, x_2, x_3\right] = ?$$

$$p = 1 \qquad s_1 = \{1,2,3\}$$

$$p = 2 \qquad s_1 = \{1\}, s_2 = \{2,3\}$$

$$s_1 = \{2\}, s_2 = \{1,3\}$$

$$s_1 = \{3\}, s_2 = \{1,2\}$$

$$p = 3 \qquad s_1 = \{1\}, s_2 = \{2\}, s_3 = \{3\}$$

Moments and Cummulants

$$Cum\left[x_1, x_2, x_3\right] = E\left\{x_1 x_2 x_3\right\} - E\left\{x_1\right\} E\left\{x_2 x_3\right\} - E\left\{x_2\right\} E\left\{x_1 x_3\right\}$$
$$- E\left\{x_3\right\} E\left\{x_1 x_2\right\} + 2E\left\{x_1\right\} E\left\{x_2\right\} E\left\{x_3\right\}$$

Calculation of Cum. of order r requires moments up to order r!

Properties of Moments and Cummulants (not complete!)

1.) $$\boxed{Cum[x_1,\ldots,x_n] = 0, \quad Mom[x_1,\ldots,x_n] \neq 0}$$

$$\{x_1,\ldots,x_\lambda\}, \{x_{\lambda+1},\ldots,x_n\} \rightarrow \psi(\omega_1,\ldots,\omega_n) =$$

stat. independent

$$\underbrace{\psi_1(\omega_1,\ldots,\omega_\lambda)} + \underbrace{\psi_2(\omega_{\lambda+1},\ldots,\omega_n)}$$

$$\frac{\partial}{\partial\omega_1 \ldots \partial\omega_n} = 0$$

Properties of Moments and Cummulants (not complete!)

2.)

$$\boxed{\begin{aligned} &Cum\left[x_1+y_1,\ldots,x_n+y_n\right]= \\ &=Cum\left[x_1,\ldots,x_n\right]+Cum\left[y_1,\ldots,y_n\right] \\ &Mom\left[x_1+y_1,\ldots,x_n+y_n\right]\neq \\ &=Mom\left[x_1,\ldots,x_n\right]+Mom\left[y_1,\ldots,y_n\right] \end{aligned}}$$

$$\{x_1,\ldots,x_n\},\{y_1,\ldots,y_n\}$$

stat. independent

Proof is similar as in 1.)

Properties of Moments and Cummulants (not complete!)

3.) $\boxed{Cum[x_1, \ldots, x_n] = 0, n > 2}$ $\{x_1, \ldots, x_n\}$ jointly Gaussian

$$\left.\begin{aligned}
\phi(\omega) &= \exp\left(j\omega m_1 - \frac{1}{2}\omega^2\sigma^2\right) \\
\psi(\omega) &= \underbrace{j\omega m_1 - \frac{1}{2}\omega^2\sigma^2} \\
&\frac{d^n}{d\omega^n} = 0,\ n > 2
\end{aligned}\right\}$$

$m_3, m_4, \ldots$

Intoduces no additional Information !

Properties of Moments and Cummulants (not complete!)

Example: Noise supression

Gaussian noise

$$z_i = y_i + x_i, \quad i = 1, 2, 3$$

$\{y_1, y_2, y_3\}$ joint prob. density function is not Gaussian !

$\{x_1, x_2, x_3\}$ joint prob. density function is Gaussian and Independent of $\{y_1, y_2, y_3\}$

with 2.) and 3.) $\longrightarrow$ $\boxed{Cum[z_1, z_2, z_3] = Cum[y_1, y_2, y_3]}$

Moments and Cummulants of Stationary Processes

$$\{X(k)\}, k = 0, \pm 1, \pm 2, \ldots$$

Real stationary random process

$$\underbrace{Mom\left[X(k), X(k+\tau_1), \ldots, X(k+\tau_{n-1})\right]}_{m_n^x(\tau_1, \ldots, \tau_{n-1})} = E\left\{X(k)\,X(k+\tau_1)\cdots X(k+\tau_{n-1})\right\}$$

$$c_n^x(\tau_1, \ldots, \tau_{n-1}) = Cum\left[X(k), X(k+\tau_1), \ldots, X(k+\tau_{n-1})\right]$$

Moments and Cummulants of Stationary Processes

1st-order cummulant

$$\boxed{c_1^x = E\left\{X(k)\right\} = m_1^x}$$

mean value

2nd-order cummulant

$$\boxed{c_2^x(\tau) = \underbrace{m_2^x(\tau)}_{\text{autocorrelation sequence}} - \left(m_1^x\right)^2}$$

covariance sequence

$$c_2^x(0) = \gamma_2^x,\ m_1^x = 0$$

variance

$$c_3^x(0,0) = \gamma_3^x,\ m_1^x = 0$$

skewness

Cummulant Spectra

$$C_n^x(\omega_1,\ldots,\omega_{n-1}) = \sum_{\tau_1=-\infty}^{\infty}\cdots\sum_{\tau_{n-1}=-\infty}^{\infty} c_n^x(\tau_1,\ldots,\tau_{n-1}) \cdot \exp\left\{-j\left(\omega_1\tau_1+\cdots+\omega_{n-1}\tau_{n-1}\right)\right\}$$

complex periodic function

Power Spectrum: n=2

$$C_2^x(\omega) = \sum_{\tau} c_2^x(\tau)\exp(-j\omega\tau) \qquad c_2^x(\tau) = \frac{1}{2\pi}\int_{-\pi}^{\pi} C_2^x(\omega)\exp(j\omega\tau)\,d\omega$$

if $\tau = 0 \longrightarrow$

$$\boxed{c_2^x(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} C_2^x(\omega)\,d\omega}$$

power

power spectrum

Linear Phase Shifts

$$Y(k) = X(k-D)$$

$$\boxed{c_n^y(\tau_1,\ldots,\tau_{n-1})} = Cum\left[Y(k), Y(k+\tau_1),\ldots,Y(k+\tau_{n-1})\right]$$

$$= Cum\left[X(k-D), X(k-D+\tau_1),\ldots,X(k-D+\tau_{n-1})\right]$$

$$\boxed{= c_n^x(\tau_1,\ldots,\tau_{n-1})}$$

Cummulant spectra supress linear phase shifts!

$\downarrow$

Cross-cummulants

Linear Phase Shifts

$$
\begin{aligned}
c_{xyx\ldots x}\left(\tau_1,\ldots,\tau_{n-1}\right) &= Cum\left[X\left(k\right),Y\left(k+\tau_1\right),X\left(k+\tau_2\right),\ldots,X\left(k+\tau_{n-1}\right)\right] \\
&= Cum\left[X\left(k\right),X\left(k-D+\tau_1\right),X\left(k+\tau_2\right),\ldots,X\left(k+\tau_{n-1}\right)\right] \\
&= c_n^x\left(\tau_1-D,\tau_2\ldots,\tau_{n-1}\right)
\end{aligned}
$$

$$
\downarrow
$$

$$
\boxed{C_{xyx\ldots x}\left(\omega_1,\ldots\omega_{n-1}\right) = C_n^x\left(\omega_1,\ldots\omega_{n-1}\right)\exp(-j\omega_1 D)}
$$

Cross-cummulant spectra preserve linear phase shifts!

$$
\downarrow
$$

e.g. for time delay estimation

Regression Coefficients

$$R(\omega) = \frac{C_{xy}(\omega)}{C_2^x(\omega)}$$

Example: linear system identification

$$Y(k) = \sum_i h(i) X(k-i), \quad E\{X(k)\} = 0$$

$$\boxed{c_{xy}(\tau)} = Cum[X(k), Y(k+\tau)] = \sum_i h(i) Cum[X(k), X(k+\tau-i)]$$

$$\boxed{= \sum_i h(i) c_2^x(\tau - i)} \longrightarrow \text{frequency-domain}$$

Regression Coefficients

$$C_{xy}(\omega) = \sum_{\tau} c_{xy}(\tau)\exp(-j\omega\tau) = \sum_{\tau}\sum_{i} h(i)\, c_2^x \underbrace{(\tau - i)}_{\text{variable change}} \exp(-j\omega\tau)$$

$$= \underbrace{\sum_{v} c_2^x(v)\exp(-j\omega v)}_{C_2^x(\omega)} \underbrace{\sum_{i} h(i)\exp(-j\omega i)}_{H(w)}$$

$$\downarrow$$

$$\boxed{R(\omega) = H(\omega)}$$

Nonminimum Phase LTI Systems

- Cummulant Spectra for n>2 preserve phase information

$$Y(k) = \sum_i h(i) X(k-i), \quad E\{X(k)\} = 0$$

$$C_n^y(\omega_1, \ldots, \omega_{n-1}) = H(\omega_1) \cdots H(\omega_{n-1}) H^*(\omega_1 + \cdots + \omega_{n-1}) \underbrace{C_n^x(\omega_1, \ldots, \omega_{n-1})}_{\gamma_n^x \, if \, \{X(k)\} is \, white}$$

Example: Bispectrum n=3

$$C_3^y(\omega_1, \omega_2) = H(\omega_1) H(\omega_2) H^*(\omega_1 + \omega_2) \gamma_3^x$$

$$\Psi_3^y(\omega_1, \omega_2) = \phi_H(\omega_1) + \phi_H(\omega_2) - \phi_H(\omega_1 + \omega_2)$$

phase of the LTI system

Nonminimum Phase LTI Systems

6th-order MA system

minimum phase system !

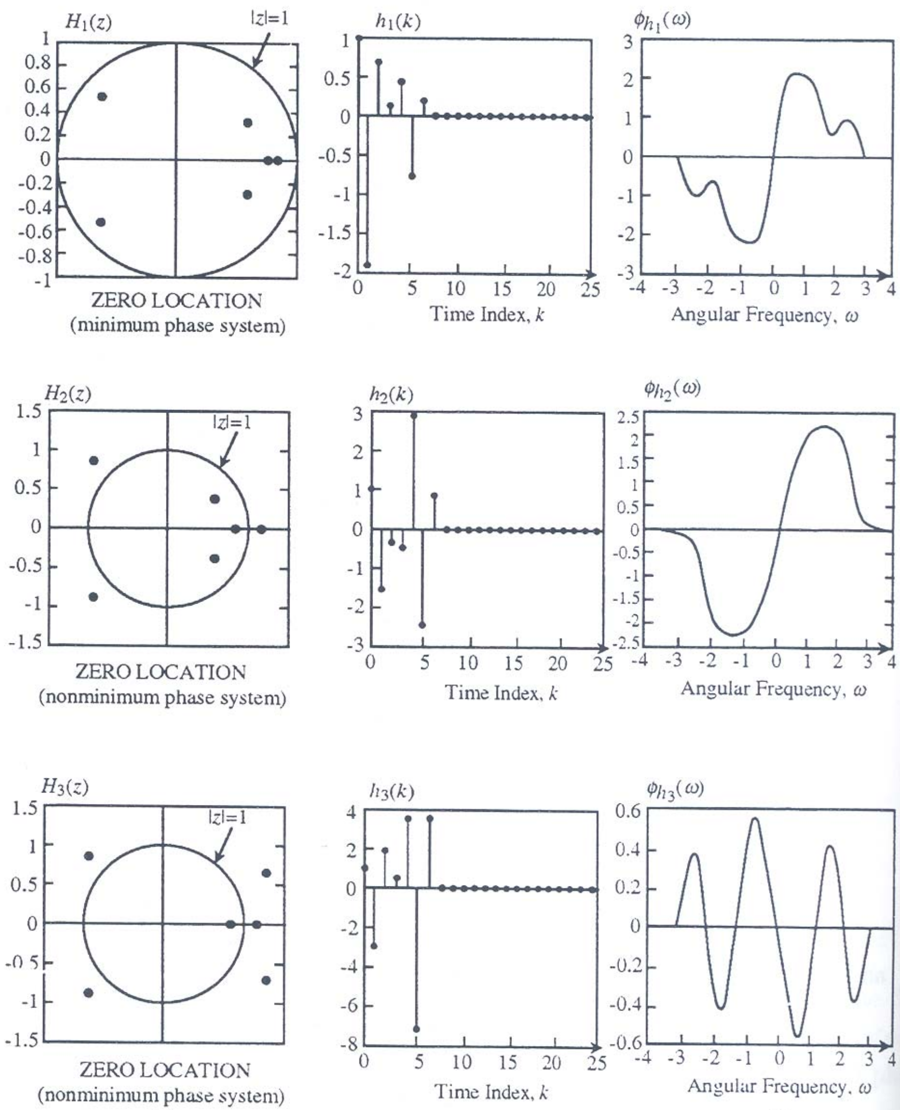
$$\boxed{\begin{aligned} H_1(z) = &\left(1-0.85z^{-1}\right)\left(1-1.2z^{-1}+0.45z^{-2}\right)\left(1-0.869z^{-1}\right)\\ &\left(1+1.1z^{-1}+0.617z^{-2}\right)\end{aligned}}$$

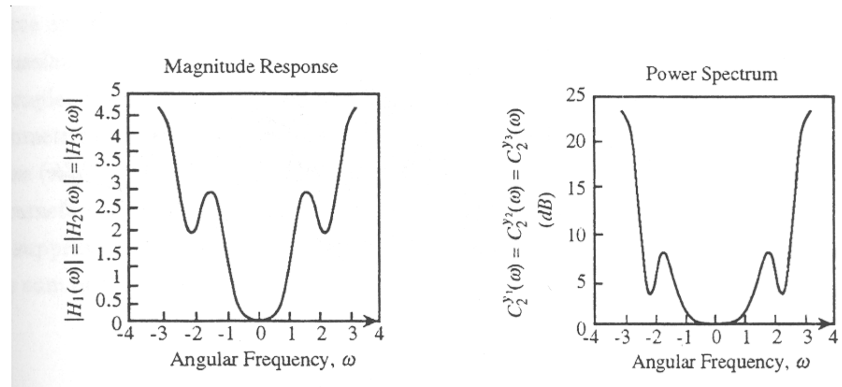
$$H_2(z) = \left(1-0.85z^{-1}\right)\left(1-1.2z^{-1}+0.45z^{-2}\right)(1-0.869z)$$
$$\left(1+1.1z+0.617z^{2}\right)$$

$$H_3(z) = \left(1-0.85z^{-1}\right)(1-1.2z+0.45z)(1-0.869z)$$
$$\left(1+1.1z+0.617z^{2}\right)$$

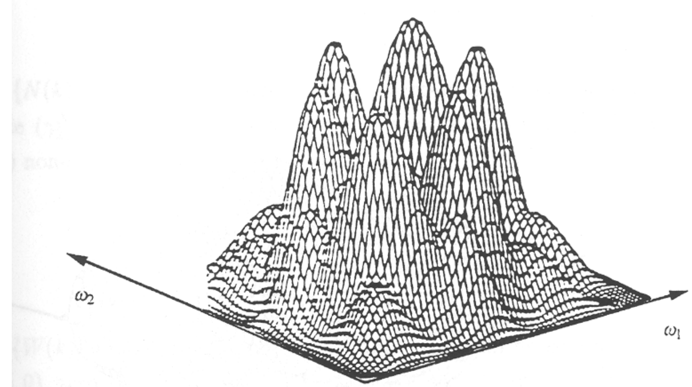
same magnitude response !

Nonminimum Phase LTI Systems





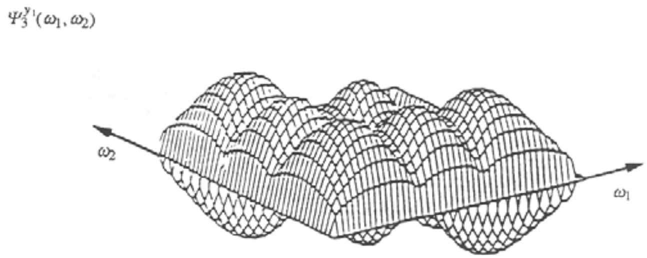
$$\left|C_3^{y_{1,2,3}}(\omega_1,\omega_2)\right| = \left|H(\omega_1)\right|\left|H(\omega_2)\right|\left|H(\omega_1+\omega_2)\right|$$



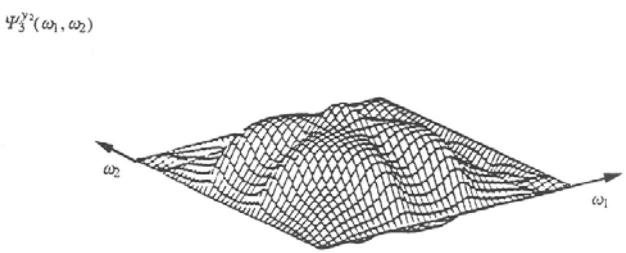
Magnitude of the Bispectrum

(e)

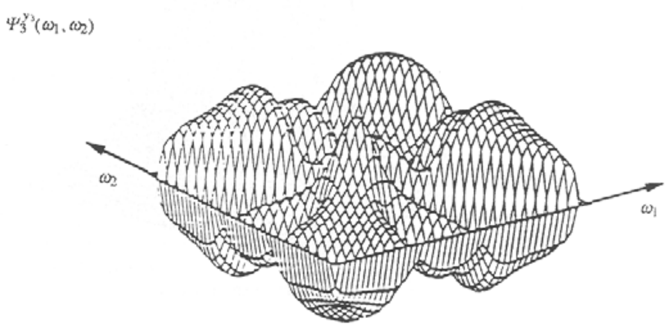
Nonminimum Phase LTI Systems



$$\Psi_3^{y_1}(\omega_1, \omega_2) = \phi_{H_1}(\omega_1) + \phi_{H_1}(\omega_2) - \phi_{H_1}(\omega_1 + \omega_2)$$



$$\Psi_3^{y_2}(\omega_1, \omega_2) = \phi_{H_2}(\omega_1) + \phi_{H_2}(\omega_2) - \phi_{H_2}(\omega_1 + \omega_2)$$



$$\Psi_3^{y_3}(\omega_1, \omega_2) = \phi_{H_3}(\omega_1) + \phi_{H_3}(\omega_2) - \phi_{H_3}(\omega_1 + \omega_2)$$

(f)

Coherency Function

$$P_n^y(\omega_1,\dots,\omega_{n-1}) = \frac{C_n^y(\omega_1,\dots,\omega_{n-1})}{\left[C_2^y(\omega_1)\cdots C_2^y(\omega_{n-1})\,C_2^y(\omega_1+\dots+\omega_{n-1})\right]^{\frac{1}{2}}}$$

Example: distinguish between linear non-Gaussian

and nonlinear processes

$$C_n^y(\omega_1,\dots,\omega_{n-1}) = H(\omega_1)\cdots H(\omega_{n-1})\,H^*(\omega_1+\dots+\omega_{n-1})\underbrace{C_n^x(\omega_1,\dots,\omega_{n-1})}_{\gamma_n^x \; if \, \{X(k)\} is \, white}$$

$$C_2^y(\omega) = \gamma_2^x \left|H(\omega)\right|^2$$

coherency index

$$\boxed{\left|P_n^y(\omega_1,\dots,\omega_n)\right| = \frac{\gamma_n^x}{\left(\gamma_2^x\right)^{\frac{n}{2}}}}$$

Coherency Function

$$\left|P_n^y\left(\omega_1,\ldots,\omega_n\right)\right| = \begin{cases} 0, & if\; Y\left(k\right) is\; Gaussian\; for\; n > 2 \\ \dfrac{\gamma_n^x}{\left(\gamma_2^x\right)^{\frac{n}{2}}}, if\;\; Y\left(k\right)\;\; is\; linear\; non-Gaussian \\ f\left(\omega_1,\ldots,\omega_n\right), ifY\left(k\right) is\; nonlinear \end{cases}$$

Nonlinear System Identification

2nd-order homogeneous Volterra system

$$Y(k) = \sum_{i_1} \sum_{i_2} h_2(i_1, i_2) X(k - i_1) X(k - i_2)$$

Gaussian

$$c_{yxx}(\tau_1, \tau_2) = Cum\left[Y(k), X(k+\tau_1), X(k+\tau_2)\right] =$$

$$= \sum_{i_1} \sum_{i_2} h_2(i_1, i_2) \underbrace{Cum\left[X(k-i_1) X(k-i_2), X(k+\tau_1), X(k+\tau_2)\right]}_{\text{Leonov-Shiryaev}}$$

Wiener

$$= \sum_{i_1} \sum_{i_2} h_2(i_1, i_2) \Big[E\left\{X(k-i_1) X(k-i_2) X(k+\tau_1) X(k+\tau_2)\right\} -$$

$$E\left\{X(k-i_1) X(k-i_2)\right\} E\left\{X(k+\tau_1) X(k+\tau_2)\right\} \Big]$$

Nonlinear System Identification

2nd-order homogeneous Volterra system

$$= \sum_{i_1}\sum_{i_2} h_2(i_1,i_2)\left[c_2^x(i_1+\tau_1)\,c_2^x(i_2+\tau_2)+c_2^x(i_1+\tau_2)\,c_2^x(i_2+\tau_1)\right]$$

$$C_{yxx}(\omega_1,\omega_2)=\sum_{\tau_1}\sum_{\tau_2} c_{yxx}(\tau_1,\tau_2)\exp\left(-j(\omega_1\tau_1+\omega_2\tau_2)\right)$$

$$\boxed{C_{yxx}(\omega_1,\omega_2)=2C_2^x(\omega_1)\,C_2^x(\omega_2)\,H_2(-\omega_1,-\omega_2)}$$

Nonlinear System Identification

2nd-order Volterra system

$$Y(k) = \underbrace{\sum_i h_1(i)\,X(k-i)}_{Y_1(k)} + \underbrace{\sum_{i_1}\sum_{i_2} h_2(i_1,i_2)\,X(k-i_1)\,X(k-i_2)}_{Y_2(k)}$$

1.)

$$c_{yx}(\tau) = Cum\left[Y(k), X(k+\tau)\right] = Cum\left[Y_1(k) + Y_2(k), X(k+\tau)\right] =$$

$$= \sum_i h_1(i)\,Cum\left[X(k-i), X(k+\tau)\right]$$

$$+ \sum_{i_1}\sum_{i_2} h_2(i_1,i_2)\,\cancel{Cum\left[X(k-i_1)\,X(k-i_2), X(k+\tau)\right]}$$

Wiener

Nonlinear System Identification

2nd-order Volterra system

$$\boxed{c_{yx}(\tau) = \sum_{i} h_1(i)\, c_2^x(i+\tau) \xrightarrow{FT} C_{yx}(\omega) = C_2^x(\omega)\, H_1(-\omega)}$$

2.)

$$c_{yxx}(\tau_1, \tau_2) = Cum\left[Y(k), X(k+\tau_1), X(k+\tau_2)\right] \longrightarrow$$

$$\sum_{i_1}\sum_{i_2} h_2(i_1, i_2)\left[c_2^x(i_1+\tau_1)\, c_2^x(i_2+\tau_2) + c_2^x(i_1+\tau_2)\, c_2^x(i_2+\tau_1)\right]$$

$$\boxed{C_{yxx}(\omega_1, \omega_2) = 2C_2^x(\omega_1)\, C_2^x(\omega_2)\, H_2(-\omega_1, -\omega_2)}$$