Optimum Detection of Deterministic and Random Signals

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***Abstract*— This paper introduces various methods for the optimum detection of deterministic and random signals in Gaussian noise. The derivation of the detectors is based on the Neyman-Pearson theorem. We will only deal with Gaussian noise and assume that the probability density function is completely known.**

***Index Terms*— Neyman-Pearson theorem, replica-correlator, matched filter, generalized matched filter, energy detector, estimator-correlator**

I. Introduction

DETECTION theory deals primarily with techniques for determing how good data obtained from a certain model corresponds to the given data set. A typical example is a radar system where the presence and absence of a target has to be determined from the measurements of a sensor array. Or in a binary communication system where the received signal has to be processed to decide whether a binary ”0” or ”1” was sent. This can certainly be extended to the case of detecting M signals. However, signal processing algorithms performing the detection process have to tackle the problem that the information-bearing signals are corrupted by noise.

In this paper we will only deal with the detection of a single signal in noise. The perhaps simplest case is when the signal is assumed to be *deterministic* (as it is for a radar system). In other cases, the signal is more appropriately modeled as a *random* process. For example the waveform of a speech signal depends strongly on the identity of the speaker, the context in which the sound is spoken etc. It is therefore not meaningful to assume the signal to be known but rather to be a random process with known covariance structure. In both cases, however, we want to find a detector which performs optimal in some sense.

II. Problem statement

The main idea behind the detection process is based on *statistical hypotheses testing*. Given an observation vector $\mathbf{x}$ and several hypotheses $\mathcal{H}_i$ (a listning of probabilistic models which may have generated the data), our aim is to find an optimal method to determine which model fits the data best. Although the number of hypotheses can in principle be arbitrary, we will only consider the case of two hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1$. We will further assume that the probability density function (PDF) for each assumed hypothesis is completely known. This leads us to the *simple* hypothesis testing problem which is formulated as

$$\begin{aligned} \mathcal{H}_0 &: x[n] = w[n] & n = 0, 1, \ldots, N-1 \\ \mathcal{H}_1 &: x[n] = s[n] + w[n] & n = 0, 1, \ldots, N-1 \end{aligned} \tag{1}$$

where $s[n]$ is the signal to be detected and $w[n]$ is a noise process. Therefore, we have to determine an *optimal* method so that for each observation data we decide whether hypothesis $\mathcal{H}_0$ or $\mathcal{H}_1$ was true. There are several mathematical approaches to solve such a problem depending on what we mean by optimal. Though, the primary approaches to simple hypothesis testing are the classical ones, namely the *Neyman-Pearson* (NP) and the *Bayesian* approach. The choice of the method depends on how much prior knowledge about the probabilities of occurence of the two hypotheses we want to incorporate in our decision process. Therefore, the problem itself determines the choice of the appropriate approach. While communication and pattern recognition systems use the Bayes risk, we will employ the NP criterion as it is the case for radar and sonar systems. Moreover, the derivation of the optimal detectors will depend on our assumption about the noise.

III. Deterministic signals

A. Replica-Correlator

We begin our discussions about optimal detection by considering the case where the signal $s[n]$ in our hypothesis testing problem (1) is deterministic. $w[n]$ is thereby assumed to be a zero mean Gaussian noise process with variance σ^2 and autocorrelation function

$$r_{ww}[k] = E\left\{w[n]w[n+k]\right\} = \sigma^2\delta[k]$$

We will refer to this as white Gaussian noise (WGN) and denote

$$\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I})$$

where $\mathbf{w} = [w[0]\ w[1]\ \ldots\ w[N-1]]^T$ is a noise vector. As already mentioned, we will use the NP criterion to derive an optimal detector. Such a detector decides hypothesis $\mathcal{H}_1$ if the

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likelihood ratio exceeds a threshold or

$$L(\mathbf{x}) = \frac{p(\mathbf{x}; \mathcal{H}_1)}{p(\mathbf{x}; \mathcal{H}_0)} \;>\; \gamma$$

where $\mathbf{x} = [x[0]\; x[1]\; \ldots\; x[N-1]]^T$ is our received data vector. Since the PDF under either hypothesis is Gaussian, i.e.

$$\begin{aligned} p(\mathbf{x}; \mathcal{H}_1) &= \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \;\exp\left[-\frac{1}{2\sigma^2}\;(\mathbf{x}-\mathbf{s})^T(\mathbf{x}-\mathbf{s})\right] \\ p(\mathbf{x}; \mathcal{H}_0) &= \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \;\exp\left[-\frac{1}{2\sigma^2}\;\mathbf{x}^T\mathbf{x}\right] \end{aligned} \tag{2}$$

we have

$$L(\mathbf{x}) = \exp\left[-\frac{1}{2\sigma^2}\left[(\mathbf{x}-\mathbf{s})^T(\mathbf{x}-\mathbf{s}) - \mathbf{x}^T\mathbf{x}\right]\right] \;>\; \gamma$$

Taking the logarithm of both sides yields

$$l(\mathbf{x}) = \ln\, L(\mathbf{x}) = -\frac{1}{2\sigma^2}\left[(\mathbf{x}-\mathbf{s})^T(\mathbf{x}-\mathbf{s}) - \mathbf{x}^T\mathbf{x}\right] \;>\; \ln\,\gamma$$

This does not change the inequality because the logarithm is a monotonically increasing transformation and both sides will be affected equally. After performing some algebraic manipulations, we get

$$\frac{1}{\sigma^2}\;\mathbf{x}^T\mathbf{s} - \frac{1}{2\sigma^2}\;\mathbf{s}^T\mathbf{s} \;>\; \ln\,\gamma$$

The second term in this inequality represents the energy of the signal. But since the signal $s[n]$ is known, we can simply move this expression to the right side. So

$$\mathbf{x}^T\mathbf{s} \;>\; \sigma^2\,\ln\,\gamma + \frac{1}{2}\;\mathbf{s}^T\mathbf{s}$$

Hence, we get a new threshold γ' and we decide $\mathcal{H}_1$ if

$$\boxed{T(\mathbf{x}) = \mathbf{x}^T\mathbf{s} = \sum_{n=0}^{N-1} x[n]s[n] \;>\; \gamma'} \tag{3}$$

The NP detector of (3) is referred to as a *correlator* or *replica-correlator* since this expression represents a correlation process of the received data $x[n]$ with a replica of the known signal $s[n]$. As expected, it consists of a test statistic $T(\mathbf{x})$ (a function of the data) and a threshold γ', which is chosen to satisfy $P_{FA} = \alpha$ for a given α. A block diagram of the detector is given in Fig. 1.

A physically relevant interpretation of (3) comes from the theory of linear vector spaces. The quantity $\mathbf{x}^T\mathbf{s}$ is termed as the *dot product* between $\mathbf{x}$ and $\mathbf{s}$ or the *projection* of $\mathbf{x}$ onto $\mathbf{s}$. According to the Schwarz inequality, the largest value

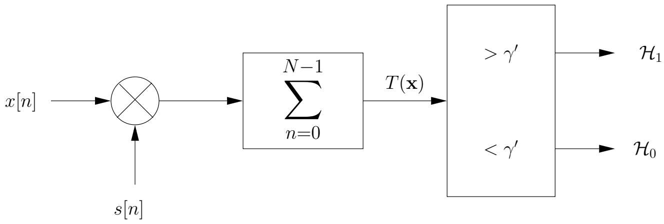

Fig. 1: Replica-correlator

occurs when the vectors are proportional to each other. More precisely, the dot product measures the similiarity between the two vectors yielding the highest value when they are parallel and the lowest value (i.e. zero) when they are orthogonal to each other. Thus, such a detector removes those components from the received data which are orthogonal to the signal. Since the noise is assumed to be independent of the signal, the detector simply eliminates it whereas the components parallel to the signal are retained.

B. *Example*

To illustrate the result above, we assume that $s[n] = A$ for some known level A, where $A > 0$. Then from (3)

$$T(\mathbf{x}) = A\sum_{n=0}^{N-1} x[n]$$

If we further divide $T(\mathbf{x})$ by NA, we decide $\mathcal{H}_1$ if

$$T'(\mathbf{x}) = \frac{1}{N}\sum_{n=0}^{N-1} x[n] = \overline{x} \;>\; \gamma''$$

But this is just the sample mean detector. Note that if $A < 0$, we get the same detector but we decide $\mathcal{H}_1$ if $\overline{x} < \gamma''$

C. *Matched Filter*

There is another important signal processing interpretation of (3). The correlation can be viewed in terms of a filtering process of the data. Since we have a summation of a finite number of samples, we take a *FIR filter* into considerations. If we now let $x[n]$ be the input to such a filter, then the output $y[n]$ at time n is given by the convolution operation, i.e.

$$y[n] = \sum_{k=0}^{n} h[n-k]x[k] \tag{4}$$

where the impulse response $h[n]$ of the FIR filter is nonzero for $n = 0, 1, \ldots, N-1$. Note that for $n < 0$ the output is zero since we assume $x[n]$ is also nonzero only over the interval $[0, N-1]$. The question that further arises is how we have to choose $h[n]$ in (4) to get the test statistic in (3). The proper choice of the impulse response is the ”flipped around” version

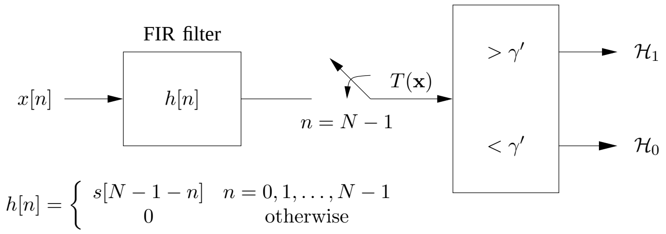

Fig. 2: Matched filter

of the signal or

$$h[n] = s[N-1-n] \qquad n = 0, 1, \ldots, N-1 \tag{5}$$

Inserting (5) into (4) and sampling the output of the FIR filter at time $n = N-1$ yields

$$\boxed{y[N-1] = \sum_{k=0}^{N-1} s[k]x[k] := T(\mathbf{x})}$$

which with a change of the summation variable is identical to the replica-correlator of (3). This implementation is shown in Fig. 2 and known as a *matched filter* because the observations are passed through a filter whose impulse-response *matches* that of the signal being sought. The output of the matched filter is sampled exactly at the moment when all observations fall within the filter's memory.

D. Properties of a Matched Filter

The matched filter may also be viewed in the frequency domain. Taking the discrete-time Fourier transform (DTFT) of (5) yields

$$H(e^{j\omega}) = S^*(e^{j\omega})e^{-j\omega(N-1)} \tag{6}$$

When we take the absolute value of this equation, the exponential term vanishes and it becomes apparent that the matched filter emphasizes the bands where there is more signal power. This can also be shown by considering the convolution sum of (4). We may view this equation as an inverse DTFT of the product of the input spectrum with the frequency response of the filter. Together with (6), we get

$$\begin{aligned} y[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\omega})e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})X(e^{j\omega})e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} S^*(e^{j\omega})X(e^{j\omega})e^{j\omega(n-(N-1))}d\omega \end{aligned}$$

Sampling the output at $n = N-1$ produces

$$y[N-1] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S^*(e^{j\omega})X(e^{j\omega})d\omega$$

This equation relates the output value of the matched filter to the spectrum of the input signal. Note that when the noise is absent or $X(e^{j\omega}) = S(e^{j\omega})$, the output becomes simply the signal energy. This result can also be seen from (3) when $x[n] = s[n]$.

Another property can be derived when we consider the signal-to-noise ratio (SNR) at the output of an FIR filter with arbitrary impulse response given by

$$\begin{aligned} \eta &= \frac{E^2\left\{y[N-1]; \mathcal{H}_1\right\}}{\mathrm{Var}\left\{y[N-1]; \mathcal{H}_1\right\}} \\ &= \frac{\left(\sum_{k=0}^{N-1} h[N-1-k]s[k]\right)^2}{E\left\{\left(\sum_{k=0}^{N-1} h[N-1-k]w[k]\right)^2\right\}} \\ &= \frac{\left(\mathbf{h}^T\mathbf{s}\right)^2}{E\left\{\left(\mathbf{h}^T\mathbf{w}\right)^2\right\}} = \frac{1}{\sigma^2}\frac{\left(\mathbf{h}^T\mathbf{s}\right)^2}{\mathbf{h}^T\mathbf{h}} \end{aligned}$$

where $\mathbf{s} = [s[0]\ s[1]\ \ldots\ s[N-1]]^T$, $\mathbf{h} = [h[N-1]\ h[N-2]\ \ldots\ h[0]]^T$ and $\mathbf{w} = [w[0]\ w[1]\ \ldots\ w[N-1]]^T$. By the Cauchy-Schwarz inequality, this equation is maximal iff

$$\mathbf{h} = c\mathbf{s}$$

which corresponds to our matched filter. Letting $c = 1$ we obtain the maximum output SNR given by

$$\eta_{max} = \frac{\mathbf{s}^T\mathbf{s}}{\sigma^2} = \frac{\varepsilon}{\sigma^2}$$

where ε is the signal energy. For the detection of a known signal in WGN, the NP criterion and the maximum SNR criterion lead to the matched filter. Since the NP criterion is optimal, the maximum SNR criterion also produces an optimal detector under these model assumptions. On the other hand, when we assume that we have non-Gaussian noise the matched filter is not optimal in the NP sense because the NP detector is not linear. However, the matched filter still maximizes the SNR at the output of a linear FIR filter (more generally, this is true for any linear filter, even for an IIR type).

E. Performance of a Matched Filter

To determine the detection performance of a matched filter, we consider again the derived test statistic of (3)

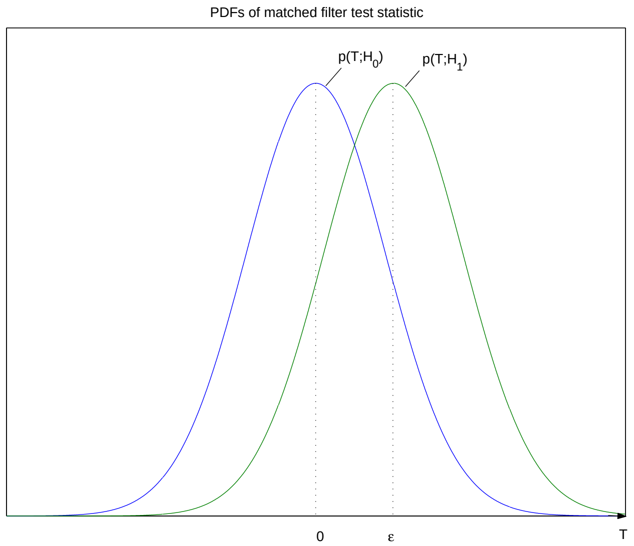

Fig. 3: PDFs of a matched filter test statistic

$$T(\mathbf{x}) = \sum_{n=0}^{N-1} x[n]s[n] = \mathbf{x}^T\mathbf{s} \ > \ \gamma'$$

We know from Chap. II that under either hypothesis $x[n]$ is Gaussian. Since $T(\mathbf{x})$ is a linear combination of Gaussian random variables, $T(\mathbf{x})$ is also Gaussian. If we compute the expected value and the variance of the test statistic (i.e. $E\{T;\mathcal{H}_i\}$ and $\mathrm{Var}\{T;\mathcal{H}_i\}$ for $i = 1, 2$), we get

$$T \sim \begin{cases} \mathcal{N}(0, \sigma^2\varepsilon) & \text{under } \mathcal{H}_0 \\ \mathcal{N}(\varepsilon, \sigma^2\varepsilon) & \text{under } \mathcal{H}_1 \end{cases} \tag{7}$$

This is the so called *mean-shifted Gauss-Gauss problem* where we decide between two hypotheses that differ by a shift in the mean of T. More precisely, the corresponding PDFs have the same shape (same variance) but are displaced by ε against each other (as illustrated in Fig. 3).

To state something about the detection performance, we divide (7) by $\sqrt{\sigma^2\varepsilon}$

$$T' \sim \begin{cases} \mathcal{N}(0, 1) & \text{under } \mathcal{H}_0 \\ \mathcal{N}(\sqrt{\varepsilon/\sigma^2}, 1) & \text{under } \mathcal{H}_1 \end{cases}$$

We see that the detection performance must increase with $\sqrt{\varepsilon/\sigma^2}$. This is obvious because increasing the energy-to-noise ratio (ENR) ε/σ^2 does not change the shape of the PDF but simply moves them further apart. To derive an expression that confirms that, we reconsider the definition for the probability of false alarm

$$\begin{aligned} P_{FA} &= P(\mathcal{H}_1; \mathcal{H}_0) = Pr\{x[0] > \gamma; \mathcal{H}_0\} \\ &= Q\left(\frac{\gamma - \mu_0}{\sigma}\right) \end{aligned} \tag{8}$$

and for the probability of detection

$$\begin{aligned} P_D &= P(\mathcal{H}_1; \mathcal{H}_1) = Pr\{x[0] > \gamma; \mathcal{H}_1\} \\ &= Q\left(\frac{\gamma - \mu_1}{\sigma}\right) \end{aligned} \tag{9}$$

where μ_0 and μ_1 are the mean values under each hypothesis and σ is the standard deviation. Using (8) and (9) in (7), we get

$$\begin{aligned} P_{FA} &= Pr\{T > \gamma'; \mathcal{H}_0\} \\ &= Q\left(\frac{\gamma'}{\sqrt{\sigma^2\varepsilon}}\right) \end{aligned}$$

$$\begin{aligned} P_D &= Pr\{T > \gamma'; \mathcal{H}_1\} \\ &= Q\left(\frac{\gamma' - \varepsilon}{\sqrt{\sigma^2\varepsilon}}\right) \end{aligned}$$

From that we can show that the detection probability is given by

$$\boxed{P_D = Q\left(Q^{-1}(P_{FA}) - \sqrt{\frac{\varepsilon}{\sigma^2}}\right)} \tag{10}$$

Since the probability of false alarm is fixed, the key parameter in (10) is the ENR. As its value increases, the argument of $Q(.)$ decreases and P_D increases. This relation is shown in Fig. 4. It is obvious that to improve the detection performance we can always increase P_{FA} and / or ε/σ^2. It is important to note that due to the ENR the detection performance stays unaffected when the shape of the signal varies. This means that two signals with arbitrary shape but same signal energy will lead to the same detection performance. We will later see that this is just the case for white Gaussian noise.

F. Generalized Matched Filter

In many cases, the assumption of white noise is not sufficient. The noise is rather modeled as *correlated* noise described by a covariance matrix $\mathbf{C}$. Thus, we now assume that

$$\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{C})$$

To determine the NP detector we use the same derivation as given in III-A. Due to the different noise assumption, the PDFs of the two hypotheses are now given by

$$\mathbf{x} \sim \begin{cases} \mathcal{N}(\mathbf{0}, \mathbf{C}) & \text{under } \mathcal{H}_0 \\ \mathcal{N}(\mathbf{s}, \mathbf{C}) & \text{under } \mathcal{H}_1 \end{cases}$$

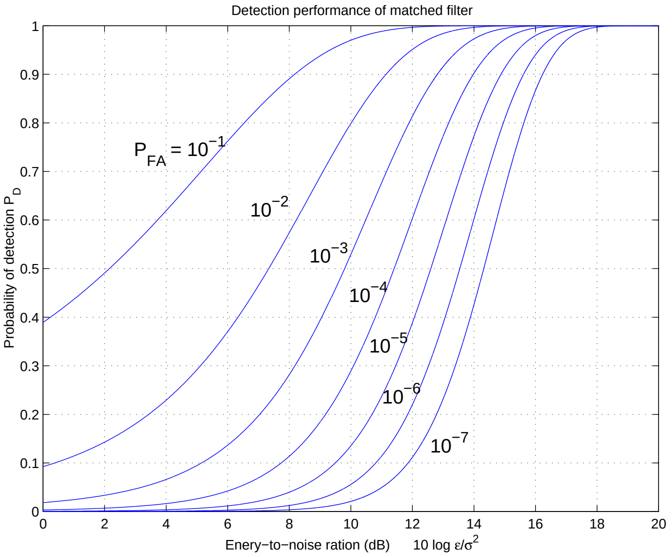

Fig. 4: Detection performance of a matched filter

or explicitly

$$p(\mathbf{x};\mathcal{H}_1) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}\det^{\frac{1}{2}}(\mathbf{C})} \exp\left[-\frac{1}{2}(\mathbf{x}-\mathbf{s})^T\mathbf{C}^{-1}(\mathbf{x}-\mathbf{s})\right]$$
$$p(\mathbf{x};\mathcal{H}_0) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}\det^{\frac{1}{2}}(\mathbf{C})} \exp\left[-\frac{1}{2}\mathbf{x}^T\mathbf{C}^{-1}\mathbf{x}\right]$$

Note that for WGN $\mathbf{C} = \sigma^2\mathbf{I}$ and the equations reduce to the ones in (2). Setting up the likelihood ratio test yields after some algebraic manipulations

$$\boxed{T(\mathbf{x}) = \mathbf{x}^T\mathbf{s}' = \sum_{n=0}^{N-1} x[n]s'[n] \ > \ \gamma'} \tag{11}$$

This detector is referred to as a *generalized replica-correlator* or *generalized matched filter* where the replica is the modified signal $\mathbf{s}' = \mathbf{C}^{-1}\mathbf{s}$. Note that in case of WGN $\mathbf{C} = \sigma^2\mathbf{I}$ and the detector reduces to the one given in (3).

G. Performance of a Generalized Matched Filter

Following the procedure, in III-E the detection performance of a generalized matched filter is given by

$$\boxed{P_D = Q\left(Q^{-1}(P_{FA}) - \sqrt{\mathbf{s}\mathbf{C}^{-1}\mathbf{s}}\right)} \tag{12}$$

The probability of detection increases monotonically with $\mathbf{s}\mathbf{C}^{-1}\mathbf{s}$ and not with the ENR ε/σ^2 as for WGN. In the latter case only the signal energy was important and not the shape. Now, the signal can be designed to maximize $\mathbf{s}\mathbf{C}^{-1}\mathbf{s}$ and therefore P_D. Note that since $\mathbf{C} = \sigma^2\mathbf{I}$ in case of WGN, (12) reduces again to its corresponding counterpart in (10).

IV. Random Signals

A. Energy Detector

As mentioned in Chap. I, there are cases in which the signal is rather modeled as a random process. Therefore, we assume that the signal $s[n]$ in our problem statement (1) is a zero-mean, white, wide-sense stationary Gaussian random process with covariance matrix $\mathbf{C}_S = \sigma_S^2\mathbf{I}$. As in the deterministic case, we will later generalize the results to a process with arbitrary covariance matrix $\mathbf{C}_S$. We could also say that the signal is WGN but the term noise is somehow inappropriate. The noise signal $w[n]$ is assumed to be white Gaussian with known variance σ^2 and to be independent of the signal.
Due to our modeling assumptions, the received data vector $\mathbf{x}$ is distributed according to a Gaussian PDF under either hypothesis but with the difference that the variance changes if the signal $s[n]$ is present (i.e. for hypothesis $\mathcal{H}_1$). Hence

$$\mathbf{x} \sim \begin{cases} \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I}) & \text{under } \mathcal{H}_0 \\ \mathcal{N}(\mathbf{0}, (\sigma_S+\sigma^2)\mathbf{I}) & \text{under } \mathcal{H}_1 \end{cases}$$

or explicitly

$$\begin{aligned} p(\mathbf{x};\mathcal{H}_0) &= \frac{1}{[2\pi(\sigma_S^2+\sigma^2)]^{\frac{N}{2}}} \exp\left[-\frac{1}{2\sigma^2}\ \mathbf{x}^T\mathbf{x}\right] \\ p(\mathbf{x};\mathcal{H}_0) &= \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[-\frac{1}{2\sigma^2}\ \mathbf{x}^T\mathbf{x}\right] \end{aligned} \tag{13}$$

After deriving the test statistic as given in III-A, we decide $\mathcal{H}_1$ if

$$\boxed{T(\mathbf{x}) = \mathbf{x}^T\mathbf{x} = \sum_{n=0}^{N-1} x^2[n] \ > \ \gamma'} \tag{14}$$

The NP detector computes the *energy* in the received data and is therefore called *energy detector*. This is intuitively clear because if the signal is present, the energy of the received data increases. This becomes clear when we consider the scaled test statistic $T'(\mathbf{x}) = (1/N)\sum_{n=0}^{N-1} x^2[n]$. Since this expression is simply the sample variance of $x[n]$, we get the the value σ^2 under $\mathcal{H}_0$ and $\sigma_S^2 + \sigma^2$ under $\mathcal{H}_1$. Note that this is in contrast to the detection of a deterministic signal where the mean value changed under either hypothesis.

B. Performance of an Energy Detector

To derive the probability of detection for a energy detector, we have to determine the PDF of the test statistic given in (14). In general, a random variable x which is the sum of the squares of N independent and identically distributed Gaussian random variables $x_i \sim \mathcal{N}(0,1)$ (i.e. $x = \sum_{i=0}^{N-1} x_i^2$) has a PDF which is *chi-squared*. Due to our modeling assumption, we have to divide the test statistic under each hypothesis

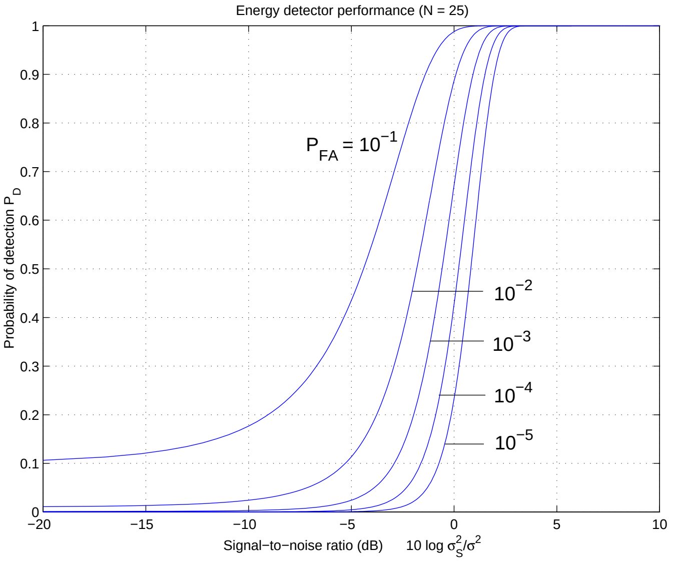

Fig. 5: Energy detector performance (N=25)

by its corresponding variance so that the result is distributed according to a chi-squared PDF

$$\begin{aligned} \frac{T(\mathbf{x})}{\sigma^2} &\sim \chi_N^2 \qquad \text{under } \mathcal{H}_0 \\ \frac{T(\mathbf{x})}{\sigma_S^2 + \sigma^2} &\sim \chi_N^2 \qquad \text{under } \mathcal{H}_1 \end{aligned}$$

After some mathematics we end up with

$$\boxed{P_D = Q_{\chi_N^2}\left(\frac{\gamma'/\sigma^2}{\sigma_S^2/\sigma^2 + 1}\right)}$$

We see that as σ_S^2/σ^2 increases, the argument of $Q_{\chi_N^2}(.)$ decreases and thus P_D increases. The probability of detection curves are illustrated in Fig. 5. The figure shows just the qualitative run of the curves and represents an approximation because the right tail probability of a chi-squared PDF is difficult to compute. However, note that even signals with lower variance than the noise variance (the signal-to-noise ratio becomes negative) can be detected given a certain P_{FA}.

C. *Estimator-Correlator*

After we assumed the signal to be a WGN process, we now generalize it to signals with arbitrary covariance matrix $\mathbf{C}_S$. Hence

$$\mathbf{x} \sim \begin{cases} \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I}) & \text{under } \mathcal{H}_0 \\ \mathcal{N}(\mathbf{0}, \mathbf{C}_S + \sigma^2\mathbf{I}) & \text{under } \mathcal{H}_1 \end{cases}$$

Using the explicit expressions for the PDFs (like the ones in (13)), we get after some algebraic manipulations and using the matrix inversion lemma

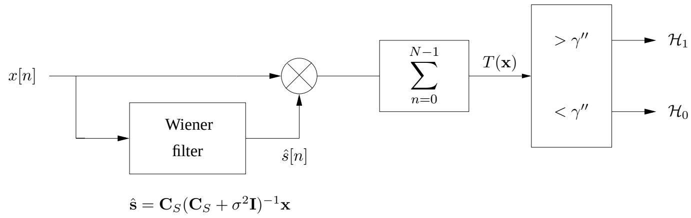

Fig. 6: Estimator-correlator for the detection of a Gaussian random signal in WGN

$$\boxed{T(\mathbf{x}) = \mathbf{x}^T\hat{\mathbf{s}} = \sum_{n=0}^{N-1} x[n]s[n] \;>\; \gamma''} \tag{15}$$

where

$$\hat{\mathbf{s}} = \mathbf{C}_S(\mathbf{C}_S + \sigma^2\mathbf{I})^{-1}\mathbf{x}$$

Since the detector correlates the received data vector $\mathbf{x}$ with an *estimate* $\hat{\mathbf{s}}$ of the signal, it is denoted as *estimator-correlator*. Note that the test statistic in (15) is a quadratic form in the data (i.e. an expression of the form $\mathbf{x}^T\mathbf{A}\mathbf{x}$ where $\mathbf{A}$ is a matrix) and thus will not be a Gaussian random variable. Recall that the energy detector was a scaled χ_N^2 random variable. Fig. 6 shows the implementation of the estimator-correlator. Note further that the estimated signal is produced by filtering the received data with a Wiener filter. Although not shown here, $\hat{\mathbf{s}}$ is therefore called a *Wiener filter estimator* of the signal.

V. Summary

We have introduced various methods for the optimum detection of deterministic and random signals in Gaussian noise. All derivations of the detectors were based on the Neyman-Pearson criterion. For deterministic signals, the replica-correlator is an optimal detector in case of white Gaussian noise. The matched filter is equivalent but simply represents another implementation of the same test statistic. To detect a known signal in colored Gaussian noise, the generalized matched filter turned out to be the optimal one. For random signals, the energy detector is an optimal detector for a zero mean, white Gaussian signal in white Gaussian noise. Generalization to signals with arbitrary covariance matrices led to the estimator-correlator. The detection of a random signal in colored Gaussian noise has not been considered here.

References

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