ANALYTICAL APPROXIMATIONS OF FRACTIONAL DELAYS: LAGRANGE INTERPOLATORS AND ALLPASS FILTERS

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ABSTRACT

We propose in this paper a new point of view which unifies two well known filter families for approximating ideal fractional delay filters: Lagrange Interpolator Filters (LIF) and Thiran Allpass Filters. We achieve this unification by approximating the ideal Fourier transform of the fractional delay according to two different Padé approximations: series expansions and continued fraction expansions, and by proving that both approximations correspond exactly either to the LIF family or to the allpass delay filters family. This leads to an efficient modular implementation of LIFs.

1. INTRODUCTION

The main motivation of this work is sound synthesis by use of physical models. Many studies have been undertaken on the modeling of physical systems by means of waveguide filters. These methods consist actually in simulating the propagation of acoustic waves with digital delay lines. These models are constrained to have a spatial step determined by the sampling rate. This is a serious drawback when a high spatial resolution in the geometry of the model is needed or when the length of the waveguide needs to vary. One can use digital filters for approximating the fractional delay, but length variations usually induce audible distortions because of local instabilities or modification in the filter's structure.

In this paper we propose a new point of view for approximating ideal fractional delays which leads to new interpretations for two families of well known approximations: Lagrange Interpolator Filters (LIF) and Thiran Allpass Filters. Both are based on expansions (series expansion and continued fraction expansion) of the analytic extention of the Fourier transform of the impulse response of the ideal fractional delay. In the case of Lagrange Interpolator Filters, this results in a new modular structure which is more efficient for time varying applications. In the case of the allpass filters, this leads to a new modular structure and theoretical model.

2. IDEAL FRACTIONAL DELAY

Continuous time-delay filters are continuous all-pass filters whose Laplace transform is $e^{-s\tau}$. Extension of time-delay filters to discrete filters leads, when the delay p is integer, to a basic delay line whose z-transform is z^{-p}. When the delay is not an integer, a fractional delay digital filter (FDDF) can be introduced as an ideal filter.

2.1. From Continuous to Discrete

The ideal FDDF may be defined by reference to continuous time-delay filters as described (see [1]) in equation 1, where we assume that the sampling rate is 1. From a time sequence $(x_k)_{k\in\mathbb{Z}}$ we rebuild the original band limited continuous signal, then we delay it and finally we re-sample it in order to get $(x_{k-d})_{k\in\mathbb{Z}}$.

This definition is no longer consistent when residual power remains at the Nyquist frequency. This can be understood since the phase at Nyquist frequency for any real time series is to be null. For instance, the sequence $(-1)^k_{k\in\mathbb{Z}}$ can not be time-shifted because it doesn't define a unique continuous signal. Notice that, in this case, the impulse response, the shifted cardinal sine $(\mathrm{sinc}\,(k-d))_{k\in\mathbb{Z}}$, is not absolutely summable. Consequently FDDFs are not BIBO[1] filters. We limit the acceptable input sequences to those without power at the Nyquist frequency. In this subspace of time sequences, FDDFs are consistent BIBO filters.

2.2. An Ideal Transfer Function

The transfer function $H^d(z)$ corresponding to the ideal FDDF of delay d does not exist, but the Fourier transform $H^d(e^{j\omega})$ of the ideal impulse response exists[2] and

[1]BIBO: Bounded Input, Bounded Output.
[2]The Fourier transform is defined for $\omega \neq \pm\pi$.

$$
\begin{tikzcd}
(x_k) \arrow[r, "h^d"] \arrow[d, "\text{reconstruction}"'] & (x_{k-d}) = (x_k) \otimes \operatorname{sinc}(k-d) \\
x(t) = \sum_{k \in \mathbb{Z}} x_k \operatorname{sinc}(t-k) \arrow[r, "\text{delay}"] & x(t-d) = \sum_{k \in \mathbb{Z}} x_k \operatorname{sinc}(t-d-k) \arrow[u, "\text{sampling}"']
\end{tikzcd}
\tag{1}
$$

is equal to $e^{-jd\omega}$. The analytical extention $\phi^d(z)$ of $H^d(e^{j\omega})$ plays the same role as the usual transfer function. This holomorphic function $\phi^d(z)$ is defined on the whole $\mathbb{C}$ plane, except for the negative real semi-axis which is an axis of discontinuity. We call[3] this function z^{-d}.

2.3. Analytical Approximations

It is clear that this ideal transfer function can not be approached by any classical series development since 0 is a singular value for z^{-d}. As in many DSP applications, we care much about the behaviour of the filter at low frequency rather than at high frequency. Moreover low frequencies are mapped in the z-plane into the neighborhood of 1. We shall demonstrate that the different low-frequency approximations for the ideal FDDF (expressed as maximally flat delay approximations around the null frequency) corresponds to different kind of Padé approximations of z^{-d} around $z = 1$, i.e approximations of $(1+x)^d$ around $x = 0$ with $x = z^{-1} - 1$.

3. LAGRANGE INTERPOLATORS

3.1. A Power Series Expansion

Let us consider the function $(1+x)^d$ defined over the $\mathbb{C}$ plane minus the real semi-axis $]-\infty, -1]$. Since this function is analytic on its definition domain, it accepts a power series expansion (eq. 2).

$$
\forall |x| < 1,\ (1+x)^d = \sum_{k=0}^{+\infty} \frac{d(d-1)\ldots(d-k+1)}{k!} x^k \tag{2}
$$

3.2. FIR Filters

From the partial series expansion of $(1+x)^d$ we derive a family of FIR filters whose transfer function $H_N^d(z)$ is defined in eq. (3) by replacing x by $z^{-1} - 1$ in eq. (2):

$$
H_N^d(z) = \sum_{k=0}^{N} \frac{d(d-1)\ldots(d-k+1)}{k!} (z^{-1} - 1)^k \tag{3}
$$

From the convergence disk of the power series expansion, we deduce that the family of filters converges at low frequency (i.e. $|\omega| < \frac{\pi}{6}$), but we will see in the following sections that this limit is usually not a key point. The transfer functions $H_N^d(z)$ may also be recast as a nested expression corresponding to the Horner scheme (eq. 4).

3.3. Link to Lagrange Interpolators

A Lagrange Interpolator Filter (LIF) of order N and delay d consists in a FIR filter, the coefficients of which are polynomials of order N in d. A LIF has to correspond to an exact delay filter when d is integer. As pointed out in [2], it appears that any FIR filter which verifies the so called maximally flat condition is a LIF. Said differently it means that a LIF corresponds to a FIR filter, the Fourier transform of which best fits the ideal Fourier transform at zero frequency.

We shall notice that each partial series expansion of z^{-d} corresponds to the best polynomial approximation around $z = 1$. Thus each $H_N^d(z)$ is the transfer function of the FIR filter, the Fourier transform of which best fits the ideal Fourier transform at zero frequency. This proves, by using the maximally flat condition, that $H_N^d(z)$ is the transfer function of the LIF of order N.

LIFs are usually only used in their optimal delay range [3] which is $d \in \left]\frac{N-1}{2}, \frac{N+1}{2}\right[$. We state a conjecture (eq. 5) about the convergence domain when

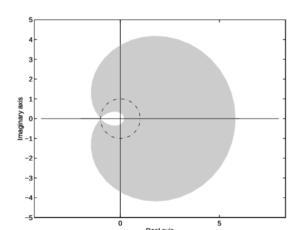

Figure 1: Unit circle inside the convergence domain of LIF.

[3] The notation z^{-d} is not really valid because $z^{-a}z^{-b} = z^{-(a+b)}$ doesn't hold for every $z \in \mathbb{C}$. However for a fixed a and b, this relation is valid for any z in the neighborhood of 1. This should be enough in our case.

$$H_{N+1}^{d}(z) = 1 + d(z^{-1} - 1)\left(1 + \frac{d-1}{2}(z^{-1} - 1)\left(\ldots\left(1 + \frac{d-N}{N+1}(z^{-1} - 1)\right)\ldots\right)\right) \tag{4}$$

$$\forall z,\ \frac{1}{2}\left|\frac{z+z^{-1}}{2} - 1\right| < 1,\ d \in]0,1[,\ \left\{\begin{array}{lcl} \lim\limits_{N\to+\infty} z^{N}.H_{2N+1}^{N+d}(z) & = & z^{-d} \\ \lim\limits_{N\to+\infty} z^{N}.H_{2N+2}^{N+d+\frac{1}{2}}(z) & = & z^{-d-\frac{1}{2}} \end{array}\right. \tag{5}$$

LIFs are used in this range. The convergence domain, $\left|\frac{z+z^{-1}}{2} - 1\right| < 2$, is geometrically characterized by the inside of the curve (fig. 1) which parametric equations are given below (eq. 6)):

$$\left\{\begin{array}{l} x = 1 + \left(2\sqrt{2} + 4\cos\theta\right)\cos\theta \\ y = \left(2\sqrt{2} + 4\cos\theta\right)\sin\theta \end{array}\right. \tag{6}$$

As we can see on figure 1, the whole unit circle is contained inside the convergence domain, except for the singular point -1. This proves that an increase of the filter order improves the approximation at any frequency.

4. ALLPASS FILTERS

4.1. A Continued Fraction Expansion

In this section, we shall use the Abramowitz notation [4] for the continued fractions expansion. From [5], p.343, we deduce a slightly simplified continued fraction expansion for $(1+x)^d$ (eq. 7, next page). This development is valid exterior to the cut along the real axis from -1 to $-\infty$.

4.2. ARMA Filters

As in 3.2, substituting x by $z^{-1} - 1$ (eq. 8 and 9, next page), we get two families of ARMA filters. Their transfer functions, $G_N^d(z)$ and $K_N^d(z)$, respectively correspond to the even and odd approximants of the given continued fraction expansion. Notice that $G_N^d(z)$ is the transfer function of an ARMA filter whose both numerator and denominator are both of degree N.

4.3. Link to Allpass Filters

On the one hand, Thiran introduced in [6] an explicit formulation for the transfer function of the allpass filter which possesses a maximally flat delay characteristic at low frequency. This consists in optimizing an allpass filter in such a way that the derivatives of the phase response up to an order N equal those of the ideal response.

But on the other hand, it appears that the transfer function $G_N^d(z)$ is a rational fraction, the numerator and denominator of which are image mirrored. This is a characteristic of allpass filter transfer functions. Furthermore, as a Padé approximation, the continued fraction $G_N^d(z)$ optimizes its first derivatives. This proves that $G_N^d(z)$ is the transfer function of the allpass delay filter of order N.

5. DISCUSSION

We have presented two new closed forms for LIF and allpass delay filters applied to fractional delay filtering.

For LIFs, the Horner scheme leads to an efficient time varying modular implementation which may decrease significantly the computing cost relatively to the Farrow structure. Figures 2 and 3 show the modular implementation of $H_3^d(z) - 1$.

It is also possible to deduce a modular implemention of a filter, the transfer function of which is given as a continued fraction expansion in z^{-1}. On figures 2 and 4, we show a way of implementing a filter which transfer function is given by the following continued fraction:

$$H(z) = \frac{a_1 z^{-1}}{g_1 -}\,\frac{b_1 z^{-1}}{h_1 -}\,\frac{a_2 z^{-1}}{g_2 -}\,\frac{b_2 z^{-1}}{h_2 -}\,\frac{a_3 z^{-1}}{g_3 -}\,\frac{b_3 z^{-1}}{h_3} \tag{10}$$

The fractional part, $1 - G_N^d(z)$ (eq. 8), is obtained after performing the following substitutions:

$$\left\{\begin{array}{lcl} z^{-1} & \to & z^{-1} - 1 \\ a_k & \to & 1 - k - d \\ b_k & \to & d - k \end{array}\right.,\quad \left\{\begin{array}{lcl} g_k & \to & 2k - 1 \\ h_k & \to & 2 \end{array}\right. \tag{11}$$

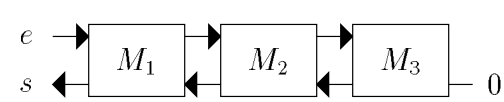

Figure 2: Modular implementation of the fractional delay filter (see figures 3 and 4 for details of modules M_k).

$$(1+x)^d = 1 + \frac{dx}{1-}\,\frac{(d-1)x}{2+}\,\frac{(d+1)x}{3-}\,\frac{(d-2)x}{2+}\,\frac{(d+2)x}{5-}\,\frac{(d-3)x}{2+}\cdots \tag{7}$$

$$G_N^d(z) = 1 + \frac{d(z^{-1}-1)}{1-}\cdots\frac{(d+N-1)(z^{-1}-1)}{(2N-1)-}\,\frac{(d-N)(z^{-1}-1)}{2} \tag{8}$$

$$K_N^d(z) = 1 + \frac{d(z^{-1}-1)}{1-}\cdots\frac{(d-N+1)(z^{-1}-1)}{2+}\,\frac{(d+N-1)(z^{-1}-1)}{(2N-1)} \tag{9}$$

In our case, unfortunately this modular implementation involves loops which include $z^{-1}-1$ instead of z^{-1}. From a computational point of view, these loops can not be samulated since the output and the input of $z^{-1}-1$ are to be inextricably linked together. Thus this modular approach seems to be usable only from a theoretical point of view.

6. CONCLUSION

Our paper has presented two new ways for deriving analytically Lagrange Interpolator Filters (LIF) and Thiran Allpass Filters from the ideal fractional delay filter. Both expressions logically lead to two fairly similar modular structures. As we have already shown in [7] it results for the LIF structure in a new robust efficient time-varying implementation. We get a new elegant theoretical closed form in the case of the allpass fractional delays.

7. REFERENCES

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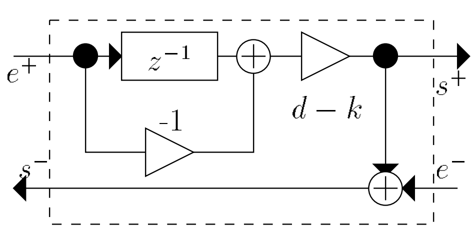

Figure 3: Module structure for the Lagrange Interpolation implementation.

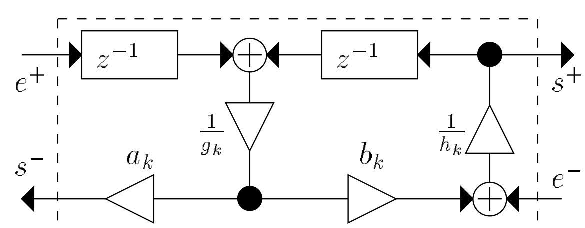

Figure 4: Module structure for the continued fraction implementation.