IMPROVED DISCRETE-TIME MODELING OF MULTI-DIMENSIONAL WAVE PROPAGATION USING THE INTERPOLATED DIGITAL WAVEGUIDE MESH

Lauri Savioja[1] *and Vesa Välimäki*[2]

[1]Lab. of Telecomm. Software and Multimedia
Helsinki University of Technology
Otakaari 1, FIN-02150 Espoo, Finland
Lauri.Savioja@hut.fi

[2]Lab. of Acoustics and Audio Signal Processing
Helsinki University of Technology
Otakaari 5A, FIN-02150 Espoo, Finland
Vesa.Valimaki@hut.fi

ABSTRACT

The digital waveguide mesh is an extension of the one-dimensional digital waveguide technique. Waveguide meshes are used for simulation of two- and three-dimensional wave propagation in musical instruments and acoustic spaces. The original waveguide mesh algorithm suffers from direction-dependent dispersion. In this paper we show that this problem may be reduced by using an interpolated rectilinear mesh. In the analysis part we show the analytical solution for the wave propagation speed and numerical simulations of the magnitude response and phase speed in both the original and the interpolated two-dimensional waveguide mesh algorithms. We demonstrate by simulation that the wave propagation characteristics of the proposed interpolated waveguide mesh are independent of direction and thus the remaining errors caused by dispersion may be corrected with a postprocessor.

1. INTRODUCTION

One-dimensional digital waveguides are a discrete numerical method widely used to model musical instruments, such as string and wind instruments [1]. Two-dimensional (2D) and three-dimensional (3D) extensions of digital waveguides have been proposed for simulation of plates and drums [2][3], and also for simulation of room acoustics [4]. Some results of using 3D waveguide meshes in simulation of the low-frequency behaviour of a listening room are presented in [5].

In the original multi-dimensional waveguide mesh, the wave propagation speed and magnitude response are functions of direction. The error increases with frequency making the technique useful at low frequencies only. One way to avoid this problem is to use other than rectilinear mesh, such as a tetrahedral or triangular mesh [6][7][8]. In an earlier study we showed that the wave propagation speed as well as the magnitude response can be made independent of propagation direction using an interpolation technique, that allows the wave to travel in 4*N* directions in an *N*-dimensional rectangular mesh [9]. In this paper we show that this scheme can be extended to allow an arbitrary number of propagation directions, although for low-order interpolation the 4*N* directions give the best results. In earlier studies interpolation techniques have been shown to be useful for one-dimensional digital waveguides [10][11].

This paper is organized as follows. In section 2 we explain the basics of multi-dimensional waveguide modeling. Section 3 describes the interpolated two-dimensional mesh. In sections 4 and 5 we compare the wave propagation speed and magnitude response of the new interpolated mesh with the original one. Section 6 concludes the paper.

2. MULTI-DIMENSIONAL WAVEGUIDE MESH

A multi-dimensional rectilinear waveguide mesh is a regular array of 1D digital waveguides arranged along each perpendicular dimension, interconnected at their crossings. Two conditions must be satisfied at a lossless junction connecting lines of equal impedance: (1) the sum of inputs equals the sum of outputs (flows add to zero) and (2) the signals in each crossing waveguide are equal at the junction (continuity of impedances). Based on these a difference equation can be derived for the nodes of an *N*-dimensional rectilinear mesh:

$$p_c(n) = \frac{1}{N}\sum_{l=1}^{2N} p_l(n-1) - p_c(n-2) \tag{1}$$

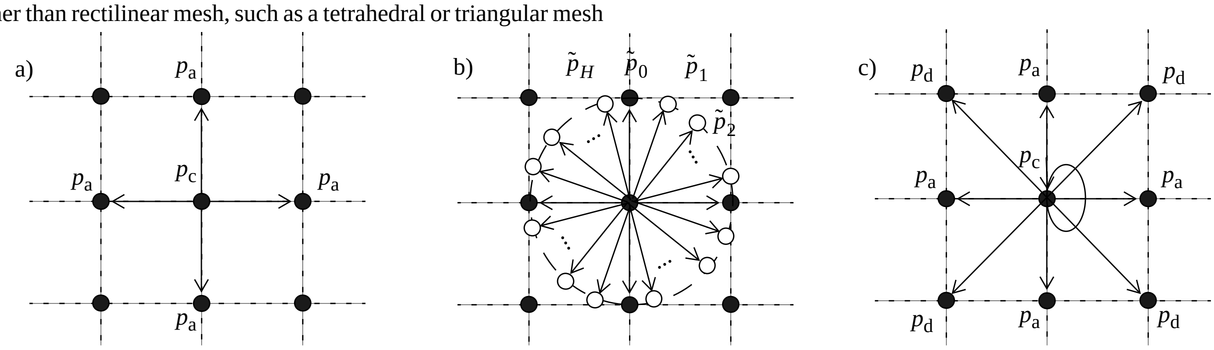

Figure 1: 2D waveguide mesh structures: a) the original mesh with 4 directions, b) the hypothetical version with *H* propagation directions, c) the new deinterpolated waveguide mesh, where the new $\tilde{p}_h$ nodes are spread onto the neighboring nodes by deinterpolation.

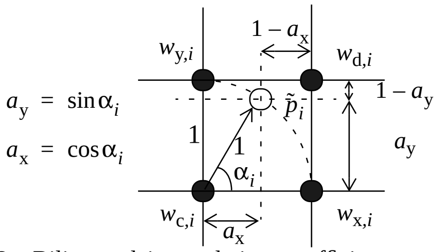

Figure 2: Bilinear deinterpolation coefficients are calculated from this geometry.

where p represents the signal pressure at a junction at time step n, subscript c denotes the junction to be calculated and index l represents its *2N* axial neighbors. The derivation of this equation and the excitation of the mesh are presented in detail in [12]. This waveguide mesh equation is equivalent to a difference equation derived from the wave equation by discretizing time and space with the forward-time-center-space method [13].

3. INTERPOLATED MULTI-DIMENSIONAL WAVEGUIDE MESH

Ideally, waves should propagate at the same speed to all directions from every node of the waveguide mesh. In the original waveguide mesh, however, waves are allowed to travel in the *2N* axial directions only. This approximation causes inaccuracies in both the magnitude response and the wave propagation speed.

In the 2D waveguide algorithm each node in the mesh has four neighbors, two on both axes (points p_a in figure 1a). These are connected by unit delay elements. To extend this scheme to arbitrary number of neighbors we add unit delay lines from a node also to other directions (hypothetical points labelled $\tilde{p}_0 \dots \tilde{p}_{H-1}$ in figure 1b). In this paper we study only symmetrical cases, where hypothetical nodes are equispaced and all the 2N axial nodes belong to $\tilde{p}_0 \dots \tilde{p}_{H-1}$. The equation that governs this situation is in principle similar to equation (1). This structure is however purely theoretical since most of the hypothetical nodes are none of the mesh nodes. Therefore other than the axial directions must have some special treatment.

3.1. Bilinear deinterpolation

To implement unit delays in diagonal directions the signal may be spread onto the nodes which are closest to the hypothetical points, as illustrated in figure 1c. This technique is called *deinterpolation* [14][10], since it is an inverse operation to interpolation. Note that the terms 'inverse interpolation' and 'decimation' have been reserved for other uses in mathematics and digital signal processing. This is why a new name had to be invented for this method.

The deinterpolation coefficients are the same as those used for interpolation. Any interpolation technique can be used [11], but we have chosen the first-order Lagrange interpolation, or linear interpolation, for its simplicity. In 2D it is called bilinear interpolation [15], since it is linear with respect to two variables. Figure 2 represents the geometry used in calculation of the bilinear interpolation coefficients:

$$
\begin{aligned}
w_{y,i} &= (1-a_x)\cdot a_y & w_{d,i} &= a_x \cdot a_y \\
w_{c,i} &= (1-a_x)\cdot(1-a_y) & w_{x,i} &= a_x \cdot (1-a_y)
\end{aligned}
\tag{2}
$$

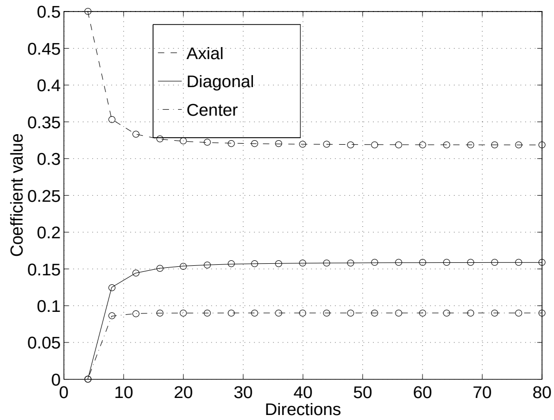

Figure 3: Weighting coefficients h_d, h_a, h_c as function of number of wave propagation directions.

3.2. Point-spreading function

When all the hypothetical nodes are deinterpolated and their contribution is added to the first part of equation (1), we obtain the difference equation for the new bilinearly deinterpolated 2D waveguide mesh:

$$
p_c(n) = \frac{2}{H}\sum_{l=1}^{3}\sum_{k=1}^{3} h_{l,k}\cdot p_{l,k}(n-1) - p_c(n-2) \tag{3}
$$

where $p_{l,k}$ represents p_c and all its neighbors p_a and p_d, and $h_{l,k}$ are the weighting coefficients of each node. Due to symmetry $h_d=h_{11}=h_{13}=h_{31}=h_{33}$, $h_a=h_{12}=h_{21}=h_{32}=h_{23}$, $h_c=h_{22}$, with

$$
\begin{gathered}
h_d = \sum_{i=0}^{H/4-1} w_{d,i} \qquad h_c = 4\sum_{i=0}^{H/4-1} w_{c,i} \\
h_a = 2\sum_{i=0}^{H/4-1} w_{x,i} = 2\sum_{i=0}^{H/4-1} w_{y,i}
\end{gathered}
\tag{4}
$$

The part of equation (1) that operates with the sum of neighbors can be thought of as a point spreading function (PSF) in 2D signal processing. It determines how an impulse propagates in different directions. In general a PSF can be thought of as a two-dimensional impulse response. Bilinear PSF's have been studied in image processing literature [15]. In the deinterpolated version the only difference between equations (1) and (3) is in the PSFs. Those equations can also be expressed by means of a 2D convolution, where the convolution kernel is one of the following 3×3 matrices:

$$
\boldsymbol{h}_{orig} = \frac{1}{2}\cdot\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix} \qquad \boldsymbol{h}_{deint} = \frac{2}{H}\cdot\begin{bmatrix}h_d & h_a & h_d\\ h_a & h_c & h_a\\ h_d & h_a & h_d\end{bmatrix} \tag{5}
$$

where $\boldsymbol{h}_{orig}$ represents the original algorithm and $\boldsymbol{h}_{deint}$ the deinterpolated one. The different scaling factors are due to the added number of wave propagation directions. The weighting coefficients as a function of number of propagation directions is illustrated in figure 3, which shows that the coefficients are the same when there are over 30 hypothetical directions. In practice the 8-directional mesh structure is the most useful one since its wave propagation characteristics are most circular symmetric.

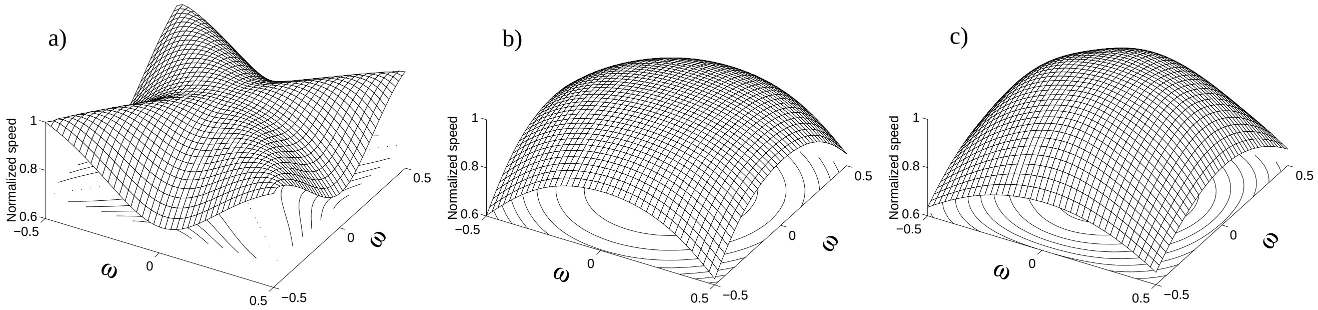

Figure 4: Normalized wave travel speeds in a) the original, b) the hypothetical 8-directional, and c) the deinterpolated 8-directional waveguide mesh structures as a function of normalized frequency.

4. WAVE PROPAGATION SPEED

The dispersion error of the original 2D waveguide mesh has been analyzed by Van Duyne and Smith [2]. The same analysis method is applied here to the new deinterpolated structure. The main principle in the analysis is the two-dimensional discrete-time Fourier transform of the difference scheme with sampling interval *T*. In the transform we take $x \leftrightarrow \xi_1$ and $y \leftrightarrow \xi_2$, so that the point (ξ_1, ξ_2) in the two-dimensional frequency space corresponds to the spatial frequency $\xi = \sqrt{\xi_1^2 + \xi_2^2}$.

4.1. The original 2D waveguide mesh

For the original difference scheme the Fourier transform is as presented by Van Duyne and Smith [2]:

$$P_{\xi_1 T, \xi_2 T}(n+1) + P_{\xi_1 T, \xi_2 T}(n-1) = 0.5 P_{\xi_1 T, \xi_2 T}(n)(e^{j\xi_1 T} + e^{j\xi_2 T} + e^{-j\xi_1 T} + e^{-j\xi_2 T}) \tag{6}$$

The desired wave propagation speed in the mesh is $c = 1/\sqrt{2}$, so that waves propagate one diagonal unit in two time steps. The ratio of the actual speed to the desired speed is

$$\frac{c'(\xi_1, \xi_2)}{c} = \frac{\sqrt{2}}{\xi T} \cdot \arctan \frac{\sqrt{4 - b^2}}{b} \tag{7}$$

where *b* is

$$b = \cos\xi_1 T + \cos\xi_2 T \tag{8}$$

4.2. Mesh with arbitrary number of directions

Although it is not possible to implement the hypothetical structure with *H* directions in the time domain, the Fourier transform may be computed, and it is

$$P_{\xi_1 T, \xi_2 T}(n+1) + P_{\xi_1 T, \xi_2 T}(n-1) = \frac{2}{H} P_{\xi_1 T, \xi_2 T}(n) \sum_{i=0}^{H-1} e^{j(\xi_1 \cos\alpha_i + \xi_2 \sin\alpha_i)T} \tag{9}$$

The equation for the speed ratio is the same as (7), where *b* is in a symmetrical case

$$b = \frac{4}{H} \sum_{i=0}^{H/2-1} \cos((\xi_1 \cos\alpha_i + \xi_2 \sin\alpha_i)T) \tag{10}$$

4.3. The deinterpolated 2D waveguide mesh

The Fourier transform for the deinterpolated structure is

$$\begin{aligned} &P_{\xi_1 T, \xi_2 T}(n+1) + P_{\xi_1 T, \xi_2 T}(n-1) = \\ &\frac{2}{H} P_{\xi_1 T, \xi_2 T}(n)[h_a(e^{j\xi_1 T} + e^{j\xi_2 T} + e^{-j\xi_1 T} + e^{-j\xi_2 T}) + \\ &h_d(e^{j\delta_+ T} + e^{j\delta_- T} + e^{-j\delta_- T} + e^{-j\delta_+ T}) + h_c] \end{aligned} \tag{11}$$

where $\delta_+ = \xi_1 + \xi_2$ and $\delta_- = \xi_1 - \xi_2$. For the speed ratio equation (7) *b* is

$$\begin{aligned} b = {} & \frac{4}{H}(h_a(\cos\xi_1 T + \cos\xi_2 T) + \\ & h_d(\cos\delta_+ T + \cos\delta_- T) + h_c/2) \end{aligned} \tag{12}$$

4.4. Stability

The difference scheme (3) is stable when *b* is real and $|b| \le 2$ since then the magnitude of the amplification factor equals to one as shown by Van Duyne and Smith [2]. In a symmetrical *H*-directional deinterpolated scheme this means that:

$$4 \cdot h_a + 4 \cdot h_d + h_c \le H \tag{13}$$

Coefficients of equation (4) satisfy the condition with the equality sign.

4.5. Comparison

Figure 4 shows the wave travel speeds in the original, the hypothetical 8-directional and the deinterpolated 8-directional structures. The frequency scale in all the figures is normalized so that the Nyquist frequency is ω_N=0.5. The useful frequency range of the original waveguide mesh is from zero to half the Nyquist frequency [2][3]. In the original algorithm there is a speed drop of over 5% in the axial direction compared to the diagonal direction at half of the Nyquist frequency, while in the deinterpolated waveguide mesh the wave propagation speed is nearly independent of direction up to the Nyquist frequency, although the speed is lower at high frequencies. Comparison of figures 4b) and 4c) shows that the bilinearly deinterpolated mesh gives a good approximation of the 8-directional mesh.

5. MAGNITUDE AND PHASE RESPONSES

Figure 5 represents the magnitude responses to different directions in both the original and the deinterpolated mesh. The results are

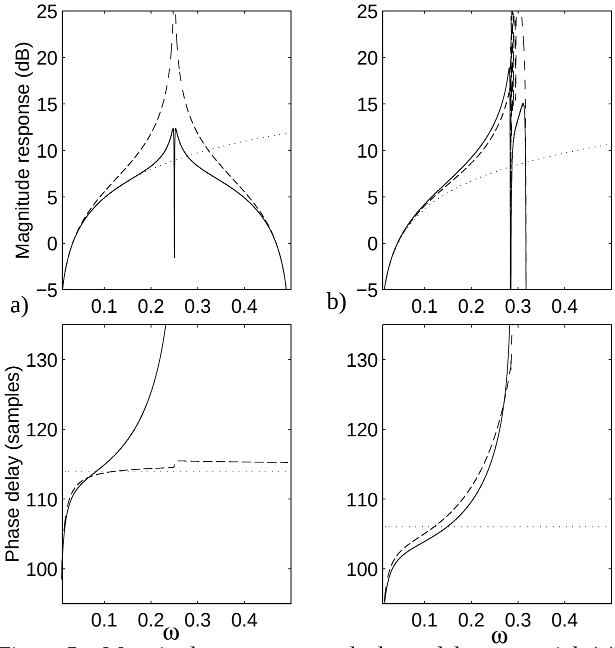

Figure 5: Magnitude responses and phase delays to axial (–) and diagonal(- -) directions in a) original, and b) deinterpolated 8-directional waveguide mesh structures. Dotted line (..) represents the ideal response in each case.

from numerical simulations with a large square mesh. The figures show that in the original mesh the axial and diagonal magnitude responses are within 1 dB from each other only up to about one fourth of the Nyquist frequency ($0.2417 \cdot \omega_N$), whereas in the deinterpolated mesh the responses are within 1 dB up to half of the Nyquist frequency ($0.4943 \cdot \omega_N$). Although the magnitude responses in figure 5b do not follow the ideal response they may be corrected by postfiltering the output. Thus the useful bandwidth is approximately doubled with the proposed algorithm.

The variation in the phase speed causes frequency distortion that may be compensated by frequency warping. The warping function may be obtained from the wave propagation speed using equations (7) and (12) and calculating the speed as function of frequency to one direction, for example one can set $\xi_2 = 0$. In a circularly symmetric case all directions give the same result and therefore any direction may be chosen when using the 8-directional deinterpolated mesh.

In practice this means that a wider frequency range can be covered in simulations with the deinterpolated mesh than with the original mesh when the mesh sizes are the same.

6. SUMMARY AND FUTURE WORK

In this paper we have shown that the wave propagation characteristics of a 2D waveguide mesh can be made independent of direction by using interpolation techniques. The best result so far was achieved by using the 8-directional bilinearly deinterpolated mesh structure. Although the characteristics are not ideal at high frequencies the distortions may be corrected by postfiltering the output. In practice this means that better quality simulations of musical instruments and room acoustics can be made with the same mesh size as before. In the future we are going to study the use of higher order interpolation which uses larger than 3×3 point spreading functions and also to extend this scheme to 3D spaces.

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