DIRECTION FINDING WITH IMPERFECT WAVEFRONT COHERENCE: A MATRIX FITTING APPROACH USING GENETIC ALGORITHM

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ABSTRACT

The performance of high-resolution direction finding methods degrades in several practical situations where the wavefronts have imperfect spatial coherence. The original solution to this problem was proposed by Paulraj and Kailath, but their technique requires *a priori* knowledge of the matrix characterizing the loss of wavefront coherence along the array aperture. Below, a novel solution to this problem is proposed, which does not require *a priori* knowledge of the spatial coherence matrix. Our technique is based on the multidimensional minimization of appropriate concentrated cost function using Genetic Algorithm (GA).

1. INTRODUCTION

High-resolution direction finding methods [1], [2] are model-based and, therefore, very sensitive to various types of model errors [2]. Usually, in direction finding algorithms each wavefront is assumed to be perfectly coherent within array aperture, i.e. its amplitude and phase are supposed to be fully correlated between any two sensors of the receiving array. Such perfect coherence of the wavefront implies that it contributes a rank-one component to the array covariance matrix. However, in many practical situations, as, for example, in sonar and radar, wavefront coherence suffers with increasing spatial separation between array sensors [3]-[7]. Such wavefront decorrelation can result from signal propagation through randomly inhomogeneous media [4]-[6], from scattering at randomly varying surfaces [6], [7], as well as from other types of stochastic model deviations. As a result, the high-resolution direction finding and detection methods are no longer applicable. In their paper [6], Paulraj and Kailath have elaborated a statistical model for sources with partial wavefront coherence and have studied how the performance of the MUSIC Direction Of Arrival (DOA) estimator degrades if spatial coherence is ignored in the signal model. They proposed the elegant technique that exploits the model developed for improving the estimation performance of MUSIC algorithm. The main drawback of their approach is the requirement of full *a priori* knowledge of spatial coherence matrix characterizing the loss of wavefront coherence along the array aperture. In practical situations, this matrix may be unknown.

Below, a new matrix fitting technique is proposed as a solution to direction finding problem in the presence of imperfectly coherent wavefronts. Unlike the Paulraj–Kailath technique [6], our algorithm does not require *a priori* knowledge of spatial coherence matrix, because the elements of this matrix are estimated jointly with signal DOA's.

2. PROBLEM FORMULATION

Consider a uniform linear array (ULA) of n sensors. Assume that there are $q < n$ narrowband stationary zero-mean mutually uncorrelated far field sources with central frequency ω_0. In this paper, we only address the source localization problem, i.e., the number of sources is assumed to be known *a priori*. First of all, consider the case of perfect wavefront coherence. The ith array vector snapshot can be modelled as [1], [2]:

$$\mathbf{r}(i) = \mathbf{A}\mathbf{s}(i) + \mathbf{n}(i) \tag{1}$$

where $\mathbf{A} = [\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_q)]$ is the $n \times q$ matrix of the wavefront vectors of each source, $\mathbf{a}(\theta) = (1, \mathrm{e}^{-j\omega_0 d \sin\theta/c}, \ldots, \mathrm{e}^{-j\omega_0 (n-1) d \sin\theta/c})^T$ is the $n \times 1$ wavefront vector corresponding to the direction θ, $\{\theta_l\}_{l=1,2,\ldots,q}$ are signal DOA's, $\mathbf{s}(i)$ is the $q \times 1$ vector of random source waveforms, $\mathbf{n}(i)$ is the $n \times 1$ vector of random sensor noise, d is the interelement spacing, c is the propagation speed, and $(\cdot)^T$ denotes transpose. The array covariance matrix [1], [2]

$$\mathbf{R} = \mathrm{E}\{\mathbf{r}(i)\mathbf{r}^H(i)\} = \mathbf{A}\mathbf{S}\mathbf{A}^H + \sigma^2\mathbf{I} \tag{2}$$

where $\mathbf{S}$ is the $q \times q$ covariance matrix of signal waveforms, $\mathbf{I}$ is the $n \times n$ identity matrix, σ^2 is the noise variance, $\mathrm{E}\{\cdot\}$ and $(\cdot)^H$ denote the statistical expectation operator and the Hermitian transpose, respectively.

Assume now that the wavefronts have imperfect coherence within the array aperture and revisit the underlying model [6]. Wavefront perturbations can be represented as multiplicative noise, i.e. the ith snapshot can be modelled as:

$$\mathbf{f}(i) = (\mathbf{G}(i) \odot \mathbf{A})\mathbf{s}(i) + \mathbf{n}(i) \tag{3}$$

where $\mathbf{G}(i)$ is the $n \times q$ matrix of random wavefront perturbations, and $\odot$ denotes the Schur-Hadamard (element by element) matrix product. The elements of matrix $\mathbf{G}(i)$ describe the amplitude and phase fluctuations of wavefronts, i.e., $[\mathbf{G}(i)]_{lk} = \zeta_{lk}(i) e^{j\phi_{lk}(i)}$. It should be noted that unlike (1), the vector process (3) is always non-Gaussian. This is the main reason why one cannot use the standard ML techniques in the situation considered.

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Following [6], assume isotropic coherency loss, i.e., consider the case where the loss across the array is the same for all wavefronts irrespective of their DOA's. Typical situations arise in long-range ocean acoustic propagation and electromagnetic propagation in the lower troposphere (see [7] and references therein). Also, this assumption may be reasonable when modelling stochastic array deviations [7].

The assumption of isotropic coherence loss means that the spatial coherence function is independent of the wavefront index k:

$$b_{lm} = \mathrm{E}\{[\mathbf{G}(i)]_{lk}[\mathbf{G}(i)]^*_{mk}\}$$
$$= \mathrm{E}\{\zeta_{lk}(i)\zeta_{mk}(i)\mathrm{e}^{j(\phi_{lk}(i)-\phi_{mk}(i))}\} \quad (4)$$

where $(\cdot)^*$ denotes the complex conjugate. From isotropic model, it follows that function (4) depends on the separation between the lth and mth sensors only, i.e., for a ULA $b_{lm} = b_{l-m}$, whereas the assumption of zero-mean phase fluctuations gives that all b_{l-m} have real values. Additionally, assume that the random wavefront perturbations, the additive sensor noises, and the source waveforms are all mutually statistically independent. Thus, the array covariance matrix for the data model (3) is given by

$$\mathbf{F} = \mathrm{E}\{\mathbf{f}(i)\mathbf{f}^H(i)\} = (\mathbf{A}\mathbf{S}\mathbf{A}^H) \odot \mathbf{B} + \sigma^2\mathbf{I} \quad (5)$$

where $[\mathbf{B}]_{lm} = b_{l-m}$ and, without loss of generality, we assume $b_0 = 1$. This normalization of $\mathbf{B}$ is equivalent to multiplying all snapshot vectors by a constant, and, obviously, it does not cause any change of the model. Therefore, $\mathbf{I} \odot \mathbf{B} = \mathbf{I}$, and (5) can be rewritten as:

$$\mathbf{F} = \mathbf{R} \odot \mathbf{B} \quad (6)$$

Thus, we conclude that $\mathbf{B}$ can be modelled as a real-valued symmetric Toeplitz positive definite matrix [6].

3. MODIFIED MUSIC

For improving the MUSIC algorithm in a situation of imperfect wavefront coherence and *a priori* known spatial coherence matrix $\mathbf{B}$, Paulraj and Kailath [6] exploited the so-called restored array covariance matrix

$$\tilde{\mathbf{R}} = \hat{\mathbf{F}} \oslash \mathbf{B} \quad (7)$$

where

$$\hat{\mathbf{F}} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{f}(i)\mathbf{f}^H(i) \quad (8)$$

is the sample estimate of the matrix $\mathbf{F}$, N is the number of snapshots, and $\oslash$ denotes the inverse of Schur-Hadamard product, i.e. $[\mathbf{C} \oslash \mathbf{B}]_{lm} = [\mathbf{C}]_{lm}/[\mathbf{B}]_{lm}$. This preprocessing operation allows to find a consistent estimate of the matrix $\mathbf{R}$. After that, the MUSIC algorithm [1] can be applied straightforwardly to the restored covariance matrix $\tilde{\mathbf{R}}$.

The main drawback of this approach is the requirement of exact *a priori* knowledge of the spatial coherence matrix $\mathbf{B}$. In practice, this assumption may be unrealistic. With imprecise knowledge of matrix $\mathbf{B}$, serious problems can occur, especially when some elements of this matrix are close to zero.

4. MATRIX FITTING TECHNIQUE

The non-Gaussian array data vector model (3) does not allow for applying the ML algorithms for direction finding in the situation of imperfect wavefront coherence. However, the natural cost function whose global minimum corresponds to the required estimates of parameters may be chosen as:

$$Z(\mathbf{\Theta}) = \| \tilde{\mathbf{R}} - \mathbf{R} \|_F^2 = \| \hat{\mathbf{F}} \oslash \mathbf{B} - \mathbf{R} \|_F^2 \quad (9)$$

where the minimization is performed over the matrices $\mathbf{R}$ and $\mathbf{B}$. The minimizer of $Z(\mathbf{\Theta})$ can be rewritten as

$$\min_{\mathbf{\Theta}} \mathrm{tr}\{(\hat{\mathbf{F}} \oslash \mathbf{B} - \mathbf{R})^2\} \quad (10)$$

which corresponds to a least-squares fit and provides a statistically consistent estimator of the $M \times 1$ vector $\mathbf{\Theta}$ of unknown parameters [7].

According to (2) and (10), we need to estimate q DOA's, q^2 real independent parameters of Hermitian matrix $\mathbf{S}$, the noise variance σ^2, and $n-1$ real independent parameters of the matrix $\mathbf{B}$. Therefore the total number of estimated parameters is $M = q(q+1)+n$. Taking into account that the Hermitian array covariance matrix is defined by n^2 real independent parameters, we have that our estimation problem is well posed if $q(q+1) \leq n(n-1)$. This is, however, always true because $q < n$.

Let us now reduce the dimension of the multidimensional search implied by (10). For fixed source DOA's and matrix $\mathbf{B}$, the optimum of (10) is achieved for

$$\hat{\mathbf{S}} = \mathbf{A}^\dagger(\hat{\mathbf{F}} \oslash \mathbf{B} - \hat{\sigma}^2\mathbf{I})\mathbf{A}^{\dagger H} \quad (11)$$

$$\hat{\sigma}^2 = \frac{1}{n-q}\,\mathrm{tr}\{\mathbf{P}_{\mathbf{A}}^\perp(\hat{\mathbf{F}} \oslash \mathbf{B})\} \quad (12)$$

$$\mathbf{A}^\dagger = (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H, \quad \mathbf{P}_{\mathbf{A}} = \mathbf{A}\mathbf{A}^\dagger, \quad \mathbf{P}_{\mathbf{A}}^\perp = \mathbf{I} - \mathbf{P}_{\mathbf{A}} \quad (13)$$

Using (2), (11), and (12), we are able to represent the minimization problem (10) in concentrated form:

$$\min_{\mathbf{\Theta}} \mathrm{tr}\Big\{\Big(\hat{\mathbf{F}}\oslash\mathbf{B}-\mathbf{P}_{\mathbf{A}}\Big(\hat{\mathbf{F}}\oslash\mathbf{B}-\frac{1}{n-q}\,\mathrm{tr}\{\mathbf{P}_{\mathbf{A}}^\perp(\hat{\mathbf{F}}\oslash\mathbf{B})\}\,\mathbf{I}\Big)\mathbf{P}_{\mathbf{A}}$$
$$-\frac{1}{n-q}\,\mathrm{tr}\{\mathbf{P}_{\mathbf{A}}^\perp(\hat{\mathbf{F}} \oslash \mathbf{B})\}\,\mathbf{I}\Big)^2\Big\}$$
$$= \min_{\mathbf{\Theta}} \mathrm{tr}\Big\{\Big(\hat{\mathbf{F}}\oslash\mathbf{B}-\mathbf{P}_{\mathbf{A}}\,(\hat{\mathbf{F}}\oslash\mathbf{B})\,\mathbf{P}_{\mathbf{A}}$$
$$-\frac{1}{n-q}\,\mathrm{tr}\{\mathbf{P}_{\mathbf{A}}^\perp(\hat{\mathbf{F}} \oslash \mathbf{B})\}\,\mathbf{P}_{\mathbf{A}}^\perp\Big)^2\Big\} \quad (14)$$

where the $(q+n-1) \times 1$ vector

$$\mathbf{\Theta} = (\mathbf{\Phi}^T, \mathbf{b}^T)^T \quad (15)$$

contains the reduced set of estimated parameters:

$$\mathbf{\Phi} = (\theta_1, \theta_2, \ldots, \theta_q)^T, \quad \mathbf{b} = (b_1, b_2, \ldots, b_{n-1})^T \quad (16)$$

If the global minimization (14) is performed then, according to (11), the final estimates of the source powers σ_l^2, $l = 1, 2, \ldots, q$ can be found as

$$\hat{\sigma}_l^2 = [\hat{\mathbf{A}}^\dagger(\hat{\mathbf{F}} \oslash \hat{\mathbf{B}} - \frac{1}{n-q}\,\mathrm{tr}\{\mathbf{P}_{\hat{\mathbf{A}}}^\perp(\hat{\mathbf{F}} \oslash \hat{\mathbf{B}})\}\,\mathbf{I})\hat{\mathbf{A}}^{\dagger H}]_{ll} \quad (17)$$

where $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ are the final estimates of the matrices $\mathbf{A}$ and $\mathbf{B}$.

Unlike the Paulraj–Kailath algorithm, our technique does not require *a priori* knowledge of spatial coherence matrix $\mathbf{B}$, because this matrix is estimated jointly with the source DOA's.

5. SIMULATIONS

In simulations, we compare DOA estimation performances of the matrix fitting technique, conventional MUSIC estimator, and Paulraj–Kailath modification of MUSIC method. We assumed a ULA with $n = 8$ sensors and half-wavelength spacing, and two mutually uncorrelated equipower signal sources impinging on the array from the directions $\theta_1 = 11°$ and $\theta_2 = 15°$. The additive Gaussian noise is uncorrelated with the sources and between array sensors and has the same variance σ^2 in each sensor. We assumed that wavefront amplitudes do not fluctuate, while the wavefront phases have Gaussian independent fluctuations with sensor-to-sensor phase increment variance σ_ϕ^2. In other words, the spatial coherence function (4) has been modelled as [4], [6], [7]:

$$b_{l-m} = \mathrm{E}\{\mathrm{e}^{j(\phi_{lk}(i)-\phi_{mk}(i))}\} = \mathrm{e}^{-\sigma_\phi^2|l-m|/2} \tag{18}$$

In all simulation examples, $\sigma_\phi^2 = 0.25$ has been taken corresponding approximately to -1.086 dB coherency loss at one-wavelength separation.

Minimization of the cost function (14) has been performed over the parameters (15), (16) using Genetic Algorithm (GA) which is known to converge to a global minimum. This algorithm seems to be suitable for solving multidimensional optimization problems in parameter estimation and array processing [8]-[10]. However, GA is known to be computationally expensive. For reduction the computational burden, the domain of variation of estimated parameters $\mathbf{b} = (b_1, b_2 \ldots, b_{n-1})^T$ has been bounded between

$$\mathbf{b}_1 = (\exp\{-\sigma_{\phi\min}^2/2\}, \ldots, \exp\{-(n-1)\sigma_{\phi\min}^2/2\})^T \quad \text{and}$$

$$\mathbf{b}_2 = (\exp\{-\sigma_{\phi\max}^2/2\}, \ldots, \exp\{-(n-1)\sigma_{\phi\max}^2/2\})^T \tag{19}$$

where $\sigma_{\phi\min}^2 = 0.09$ and $\sigma_{\phi\max}^2 = 0.49$, respectively. Similarly, the estimated DOA's have been bounded too, within the interval $6° \div 20°$. This corresponds to a very rough pre-estimation of the DOA localization sectors by conventional beamformer, which is relatively insensitive to the coherency loss compared with the high-resolution direction finding methods [3]. The following parameters of GA have been taken in simulations: number of generations = 100, number of individuals in one generation = 30, binlength = 20, probability of crosspower = 0.75, and probability of mutation = 0.001.

A total of 100 independent simulation runs have been performed to compute the experimental Root-Mean-Square Error (RMSE) of DOA estimation for each algorithm and simulated point. In all examples, the Paulraj–Kailath method has been tested in two different modes. The first one, referred to as *exact* Paulraj–Kailath method, corresponds to precise *a priori* knowledge of the coherence matrix $\mathbf{B}$. The second mode, referred to as *approximate* Paulraj–Kailath method, corresponds to the case where this matrix is known with a small error which can easily occur in practice. In the second mode we assume that the restored array covariance matrix (7) is calculated using the imprecisely known matrix $\mathbf{B}$. In turn, this matrix is calculated using the model (18) and the measured value of σ_ϕ^2, i.e. $\tilde{\sigma}_\phi^2 = 0.27$. This corresponds to $\simeq 8\%$ measurement error of σ_ϕ^2.

Fig. 1 compares experimental RMSE's of DOA estimation for conventional MUSIC, Paulraj–Kailath, and matrix fitting techniques versus the number of snapshots for the fixed Signal to Noise Ratio (SNR) equal to 20 dB for each source. Fig. 2 shows the same curves but for absolute values of DOA estimation bias. Fig. 3 demonstrates experimental RMSE's of DOA estimation for conventional MUSIC, Paulraj–Kailath, and matrix fitting techniques versus SNR for the fixed number of snapshots $N = 100$. Fig. 4 shows the same curves as in Fig. 3 but for absolute values of DOA estimation bias.

It can be seen from Figs. 1 and 2 that for high SNR the proposed matrix fitting technique significantly outperforms both conventional MUSIC and Paulraj–Kailath algorithm in the case of moderate and large number of snapshots ($N < 10^4$). Moreover, in the presence of the small measurement error of the matrix $\mathbf{B}$ the performance of Paulraj–Kailath technique degrades significantly. A surprising fact following from Figs. 1 and 2 is that for moderate and large number of snapshots (e.g., for $N \leq 3000$) the conventional MUSIC algorithm can perform better than both exact and approximate Paulraj–Kailath techniques. This fact can be explained by weak statistical consistence of the estimate (7) based on the inverse of Schur-Hadamard product.

Figs. 3 and 4 demonstrate that for moderate number of snapshots ($N = 100$) only the proposed matrix fitting technique can provide satisfactory performance for SNR ≥ 0 dB. In this situation, performances of conventional MUSIC and both exact and approximate Paulraj-Kailath techniques degrade in the whole range of SNR.

In order to evaluate relative computational loads, we compared the computational time of matrix fitting and Paulraj-Kailath technique (MUSIC spectral function has been calculated with the angular grid $0.1°$ in the whole array field of view $[-90°, 90°]$). Our comparison shows that the matrix fitting technique is more expensive in the situation considered (approximately with the factor 10). This is the payment for the improved performance.

6. CONCLUSIONS

A novel matrix fitting approach to direction finding with imperfect wavefront coherence is proposed. Unlike the well-known Paulraj–Kailath method, our algorithm does not require *a priori* knowledge of the spatial coherence matrix, because the elements of this matrix are estimated jointly with signal DOA's. Genetic Algorithm is exploited for multidimensional optimization of the appropriate concentrated cost function. Computer simulations have shown significant improvement of DOA estimation performance of the proposed technique relative to conventional MUSIC and Paulraj–Kailath methods. The payment for the improved estimation performance is higher computational complexity.

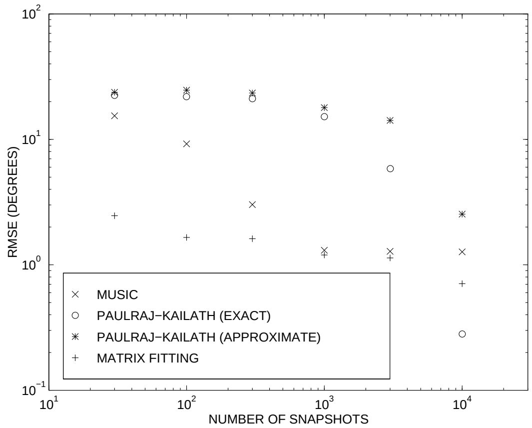

Figure 1. Experimental RMSE of DOA estimation versus the number of snapshots. $\mathrm{SNR} = 20$ **dB.**

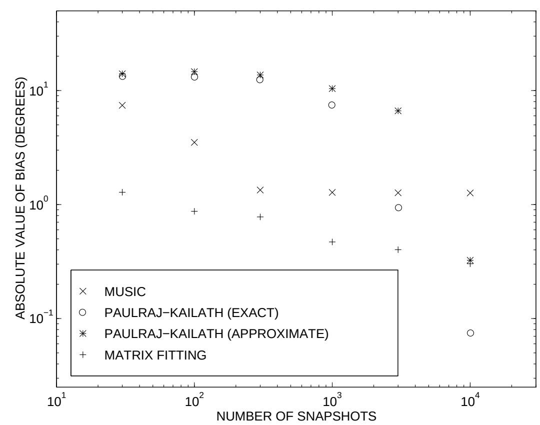

Figure 2. Experimental absolute bias of DOA estimation versus the number of snapshots. $\mathrm{SNR} = 20$ **dB.**

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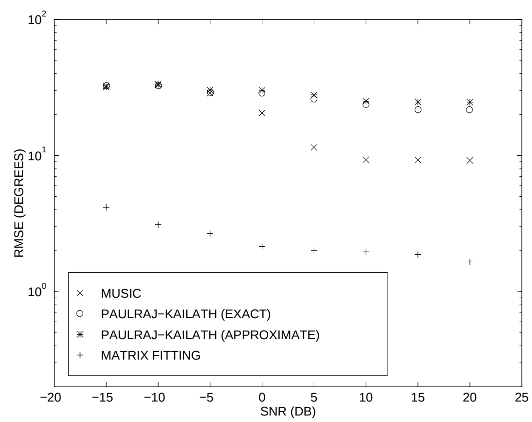

Figure 3. Experimental RMSE of DOA estimation versus SNR. The number of snapshots $N = 100$**.**

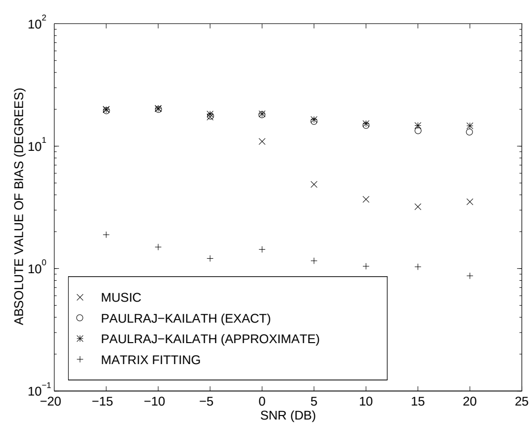

Figure 4. Experimental absolute bias of DOA estimation versus SNR. The number of snapshots $N = 100$**.**