A SUBSPACE FRAMEWORK FOR FAST PARAMETER ESTIMATION WITH KNOWN WAVEFORMS

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ABSTRACT

An efficient scheme for implementing a search of a likelihood function of known form at moderate to high SNR is constructed. Often, the original function to be searched is ill behaved with many local extreme points. By projecting the signal onto a subspace of replica waveforms we first find the maximum of a related function that is more well behaved, and then follow with a local search on the original function. The approach builds on a method of estimation of time delay of a narrowband signal [2], and it can be used to improve the efficiency of Fast Maximum Likelihood [3] estimation.

1. SIGNAL MODEL

Let us assume that a vector of received data can be modeled as a separable non-linear model corrupted by additive noise:

$$y = s(\theta)b + w \tag{1}$$

The vector s is parameterized by a vector θ. Both θ and b are assumed unknown but deterministic.

If w is zero mean complex white Gaussian noise, the maximum likelihood estimate (MLE) of the parameters is equivalent to

(a) solving the non-linear least squares problem.

$$\hat{\theta} = \arg\max |P(\theta)y|^2 \tag{2}$$

where $P(\theta) = s(s^H s)s^H$ and represents the projection onto the subspace $< s(\theta) >$.

(b) Then b is estimated by solving a linear least squares problem with $\theta = \hat{\theta}$.

The ML estimation of the vector θ of the parameter values can, therefore, be reduced to localization of the global maximum of the multidimensional function given by

$$JML(\theta) = |P(\theta)y|^2 \tag{3}$$

The implementation of the exact ML estimation scheme involves a multidimensional search which may not be practical in many circumstances.

A Fast Maximum Likelihood (FML) approach was developed in [3] that significantly reduced the number of required computations by making use of the known shape of the compressed likelihood function of formula 2 for certain important signal models (e.g., broad band HFM unknown delay and stretch). The procedure developed in [3] consisted of two primary steps:

1. localize the ridges of JML(θ) on which the global maximum is likely to be situated
2. search efficiently for the global maximum along the localized ridges

Using the initial parameter estimates obtained in step 2, a fine-grid, Newton search or any other suitable optimization procedure may be used to get final estimates. The second step in this procedure requires the "ridge" structure to have only a single local maximum.

However, in some cases, the ridge itself may be difficult to search due to local minima. Therefore, one may wish to replace it with a function that has a similar maximum value, but is more easily searched. A general framework in which this is done is by replacing each $s(\theta)$, by a $N \times K$ matrix, $Q(\theta)$. Each element of $J\hat{M}L(\theta)$ is now the magnitude of the projection onto a K dimensional subspace which spans the vector $s(\theta)$, whereas each element of $JML(\theta)$ is the projection onto the one dimensional vector $s(\theta)$. Then each element of the new function is computed as

$$J\hat{M}L(\theta) = |P(\theta)x|^2 \quad (4)$$

where $P(\theta)$ is the projection matrix onto a subspace of $Q(\theta)$.

2. ILLUSTRATION OF METHOD

The method begins by replacing each $s(\theta)$ with a set columns that span the original. That is, for each $N \times 1$ vector $s(\theta)$, we form a $N \times K$ matrix $Q(\theta)$ where $s(\theta)$ is in the span of $Q(\theta)$. The signal model for this related problem is given as

$$\hat{y} = Q(\theta)b + w \quad (5)$$

The elements of the new function are then computed as

$$J\hat{M}L(\theta) = x^H Q(\theta)(Q(\theta)^H Q(\theta))^{-1} Q(\theta)^H x \quad (6)$$

where $Q(\theta)$ is chosen such that $J\hat{M}L(\theta)$ has a maximum value close to that of $JML(\theta)$. For the purpose of implementation, the columns of $Q(\theta)$ could be orthogonalized and $J\hat{M}L_m$ can be calculated as the sum of inner products.

Figure 1 shows a geometrical interpretation where the received signal vector is being projected onto a two dimensional subspace that contains the one dimensional matching vector. To obtain the subspace projection, one could add the inner products with the orthogonal X and Y axes.

An illustrative example is given by the estimation of a time delay for a real sinusoidal burst signal given an N sample receive vector . Each matching vector $s(\tau)$ would be a delayed sinusoidal

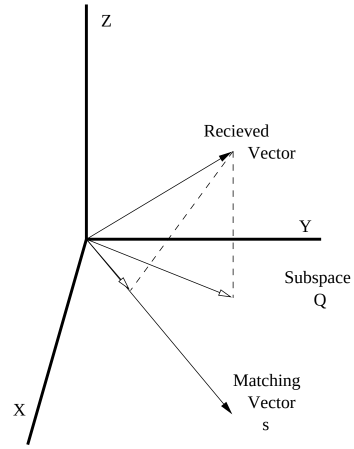

Figure 1: Projection Subspace

burst.

$$s(\tau) = \begin{bmatrix} p_{1,\tau}\sin(T-\tau) \\ p_{2,\tau}\sin(2T-\tau) \\ \vdots \\ p_{N,\tau}\sin(NT-\tau) \end{bmatrix} \quad (7)$$

where $p(nT-\tau) \equiv p_{n,\tau}$ is a sampled pulse function and T is the sampling interval. The autocorrelation function for this waveform is shown in Figure 2. The oscillations caused by the phase would make this function extremely difficult to search. Instead, for each vector s we form a matrix which consist of both a delayed cosine and sine burst and project onto this subspace.

$$Q(\tau) = \begin{bmatrix} p_{1,\tau}\sin(T-\tau) & p_{1,\tau}\cos(T-\tau) \\ p_{2,\tau}\sin(2T-\tau) & p_{2,\tau}\cos(2T-\tau) \\ \vdots & \vdots \\ p_{N,\tau}\sin(NT-\tau) & p_{N,\tau}\cos(NT-\tau) \end{bmatrix} \quad (8)$$

The result of this operation is shown in Figure 3. This can more easily be searched for an approximation to the peak and then the original autocorrelation can be searched using the previous result as a starting point.

3. BLOCK LINEAR CHIRP

In this section we seek to extend this FML approach to a problem with a compressed likelihood

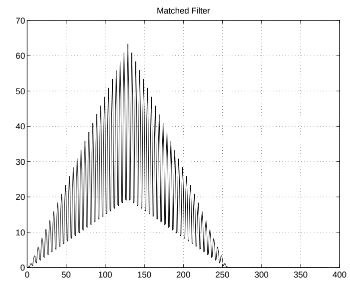

Figure 2: Standard Autocorrelation

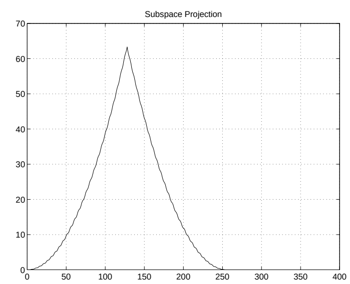

Figure 3: Subspace Projection

function that is not as well behaved. The signal considered is a linear stepped FM consisting of 16 contiguous complex sinusoids each with a distinct frequency. A time-frequency representation of this signal is given in Figure 4. The signal model 1 again applies with $\theta = (c, tdelay)$, where c = doppler stretch and tdelay is time delay.

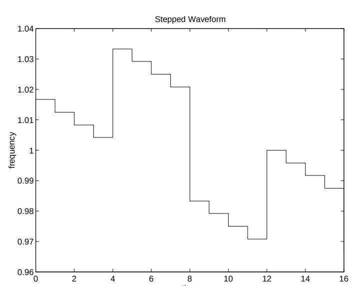

Figure 4: Waveform

The likelihood function for this signal model has a complex structure with a the loci of parameter values corresponding to the local ridge peak as shown in the left side trace in Figure 5. The corresponding maximum values along the ridge for the left trace of Figure 5 are shown by the dashed line in Figure 6. Because of the large number of local maxima this ridge function may not be easily searched and Step 2 of the FML procedure cannot be implemented in a numerically efficient way. The trace on the right of Figure 5 is from the subspace projection function discussed later.

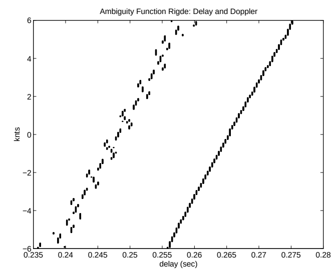

Figure 5: Ridge

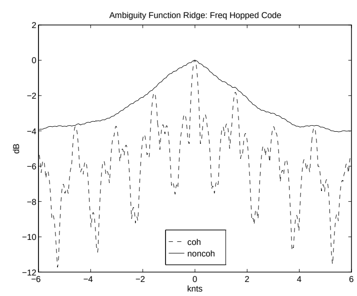

Figure 6: Ambiguity Ridge

Following the discussion in Section 2 a modified likelihood function is developed by considering $s(\theta)$ to be embedded in a larger space spanned by $s1(\theta)$,$s2(\theta)$,$s3(\theta)$,and $s4(\theta)$. Here $s1$,...$s4$ represent the 4 linear blocks of the signal. Thus rather than computing the projection on $s(\theta)$ directly we consider the modified likelihood function, in which we project on a subspace containing $s(\theta)$:

$$J\hat{M}L(\theta) = |PQ(\theta)y|^2 \tag{9}$$

where $PQ(\theta) = Q(\theta)(Q(\theta)^H Q(\theta))Q(\theta)^H$ and $Q(\theta)$ is the 4 column matrix

$$Q = \begin{bmatrix} s1 & s2 & s3 & s4 \end{bmatrix} \tag{10}$$

Since the blocks do not overlap in time, they are orthogonal and the projection operation can be performed as a non-coherent addition, that is:

$$J\hat{M}L(\theta) = |P1(\theta)x|^2 + |P2(\theta)x|^2 + |P3(\theta)x|^2 + |P4(\theta)x|^2 \quad (11)$$

where $P1(\theta) = s1(\theta)(s1(\theta)^H s1(\theta))s1(\theta)^H$ etc.

This function is well-behaved and has the ridge structure shown in Figure 5 with the corresponding values shown as the solid line in Figure 6 . The absence of strong local maxima make this function a suitable candidate for the FML procedure.

Using this modified likelihood function the performance of the proposed FML estimation approach was evaluated. Step 2 of the FML procedure, the search along the ridge to obtain an initial estimate, was performed using a golden sections search procedure. This procedure has the advantage of not requiring derivatives and of accommodating the small ripples present in the ridge of the modified likelihood function. The mean-squared error performance of the proposed estimator through this initial stage is shown in Figure 7 for the delay at various SNRs. The results are based on 400 independent trials.

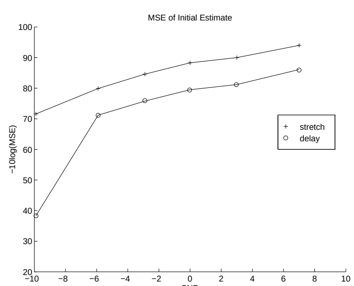

Figure 7: MSE of Initial Estimate

From these initial estimates final estimates are obtained using a fine-grid or Newton search using the original, unmodified likelihood function, $JML(\theta)$. The unmodified likelihood function offers better parameter resolution once a reasonable initial estimate is available. If Newton search is used for the final estimates, it may be necessary to start a few searches to insure that the first estimate was within the region of convergence of the global maxima. Starting points for any additional searches are chosen with knowledge of the likelihood function shape.

4. SUMMARY

We have presented a general framework which can be used to find a maximum in cases where a standard search method may be impractical and the shape of the function is known. By projecting onto a larger subspace, the function can be more efficiently searched. The result of this search can be used as a seed for a localized search on the original function.

5. REFERENCES

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