

SIGNAL DE-NOISING USING THE WAVELET TRANSFORM AND REGULARIZATION

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ABSTRACT

This paper ¹ presents a new signal de-noising algorithm using wavelets. We have developed a filtering scheme in the wavelet domain, that involves selective smoothing at each scale of the time-frequency plot. The amount of smoothing is controlled by regularizing factors, and gradient-based switches are used to avoid distortion of signal features. The algorithm is seen to compare favorably to that of Mallat et al [5, 6], as it is able to recover both the smooth portions as well as Brownian texture in the input, from the noisy signal.

1. INTRODUCTION

This paper presents a signal de-noising algorithm based on the wavelet transform. The classical Wiener-filtering for de-noising does not work well in a non-stationary environment. Wavelets naturally suggest themselves for this purpose due to their time-frequency localization properties [1]-[3]. Previous wavelet-based approaches have used statistical tests that assume the signal to be modeled well by the wavelet bases (Porat and Friedlander [4]), or have concentrated on the wavelet transform modulus maxima (Mallat and Hwang [5] and Mallat and Zhong [6]). Bertrand et al [7] consider selective reconstructions from some of the wavelet coefficients, and also consider a 'generalized thresholding' of the coefficients in a manner that mimics the Wiener filter. In Donoho and Johnstone [8] and Donoho [9], a thresholding rule is developed based on the statistics of the noise process. Ainsleigh and Chui [10] have developed an FFT-based filtering scheme in the wavelet domain, for removal of impulsive noise.

We have developed a filtering scheme in the wavelet-domain, using the entire wavelet transform, and avoiding manual selection of coefficients. We apply *selective* smoothing at each scale of the transform, and then reconstruct the de-noised version of the signal. The amount of smoothing is controlled by a regularization factor, and the smoothing is made selective using gradient-based *switches* so that important signal features are not destroyed. The selective smoothing at each scale enables us to obtain good de-noising even on a signal that has both smooth variations as well as sharp steps and Brownian texture. Simulation results are presented to illustrate the efficiency of our method.

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2. DESCRIPTION OF THE ALGORITHM

Let $\{x_n\}$ be the signal of interest and $\{y_n\}$, the observed signal, given by $y_n = x_n + w_n$ where $\{w_n\}$ is the additive noise. We do not assume any knowledge regarding the probability distribution of w_n except that w_n is zero mean. Thus, non-stationary noise assumption is permissible. Next, let $c_k^0 = y_k$, and let c_k^j be the k^{th} wavelet coefficient at resolution j , $j = 1, \dots, J$; obtained by passing $\{y_n\}$ through the subband filtering structure associated with an orthonormal wavelet. At the lowest resolution J , let the sub-sampled output of the last low-pass filter be denoted by $\{s_k^J\}$.

We then apply the following filter to $\{c_k^j\}$, $j = 1, \dots, J$ and $\{s_k^J\}$:

$$\hat{c}_k^j = h_k^j c_k^j + h_{k+1}^j c_{k+1}^j l_{k,k+1}^j + h_{k-1}^j c_{k-1}^j l_{k-1,k}^j \quad (1)$$

$$\hat{s}_k^J = H_k^J s_k^J + H_{k+1}^J s_{k+1}^J L_{k,k+1}^J + H_{k-1}^J s_{k-1}^J L_{k-1,k}^J \quad (2)$$

The de-noised version of the signal is obtained by reconstruction from the coefficients $\{\hat{c}_k^j\}$ and $\{\hat{s}_k^J\}$ instead of the original wavelet coefficients $\{c_k^j\}$ and $\{s_k^J\}$. Here $\{l_{k,k+1}^j\}$ are gradient-based *switches*, defined for each $j = 1, \dots, J$ by

$$l_{k,k+1}^j = \begin{cases} 0 & \text{if } \left| |c_k^j| - |c_{k+1}^j| \right| > t^j \\ 1 & \text{otherwise} \end{cases} \quad (3)$$

t^j are some thresholds to be selected. The switches $\{L_{k,k+1}^J\}$ are defined analogous to (3), and have a corresponding threshold T^J . The role of the switches is to avoid destroying of signal features by recognizing large changes in the wavelet coefficient amplitude and keeping them unaffected by the smoothing. The scalars $\{h_k^j\}$ and $\{H_k^J\}$ are chosen to minimize the cost function

$$B = \sum_{j=1}^J (\lambda^j \sum_k (c_k^j - \hat{c}_k^j)^2 + \sum_k l_{k-1,k}^j (\hat{c}_k^j - \hat{c}_{k-1}^j)^2) + \Lambda^J \sum_k (s_k^J - \hat{s}_k^J)^2 + \sum_k L_{k-1,k}^J (\hat{s}_k^J - \hat{s}_{k-1}^J)^2 \quad (4)$$

Here $\{\lambda^j\}$ and Λ^J are regularizing terms that control the amount of smoothing. The above minimization can be achieved by separately minimizing, for each $j = 1, \dots, J$,

$$b^j = \lambda^j \sum_k (c_k^j - \hat{c}_k^j)^2 + \sum_k l_{k-1,k}^j (\hat{c}_k^j - \hat{c}_{k-1}^j)^2 \quad (5)$$

with respect to $\{h_k^j\}$, and likewise, minimizing

$$B^J = \Lambda^J \sum_k (s_k^J - \hat{s}_k^J)^2 + \sum_k l_{k-1,k}^J (\hat{s}_k^J - \hat{s}_{k-1}^J)^2 \quad (6)$$

with respect to $\{H_k^J\}$. This was done by imposing $\frac{\partial B^J}{\partial h_k^J} = 0$ and $\frac{\partial B^J}{\partial H_k^J} = 0$ for all k . This gives rise to a banded system of equations in $\{h_k^j\}$ at each scale j , and likewise in $\{H_k^J\}$.

3. HOW THE METHOD WORKS

The wavelet transform can be observed to achieve an effect similar to de-correlation and energy compaction, so that a typical signal would often have a transform in which the essential features of the signal are captured in a single coefficient that is markedly different in value from its neighbors [8]. So, a sudden change in the wavelet coefficient magnitude is likely to be a signal feature and not a creation of the noise. Hence performing the blurring across such a sudden change would be distorting the signal features. So the switches are used in (1), (2) and (4) to prevent the blurring in these cases. The cost function (4) indicates that one is trying to smooth the signal $\{c_k^j\}$ at each scale j by reducing the power in its gradient. The amount of smoothing is controlled by the regularization parameter. Separate processing at each scale leads to finer control of the overall performance.

4. RESULTS FROM IMPLEMENTATION

Figure 1 shows the signal on which the scheme described above was tested. Note that it is similar to the one used by Mallat and Zhong [5] having isolated singularities as well as Brownian texture. Figure 2 shows the signal with zero mean Gaussian noise added to it. The algorithm described above was run on the noisy signal in Figure 2, using the Daubechies 6-tap wavelet filter, and the parameters t^j and λ^j as shown in Table 1.

j	1	2	3	4	5	6	7	8
t^j	1.3	1.0	0.9	1.6	2	2	0	0
λ^j	0.00001	0.01	0.03	5	20	40	10	10
$J=8$			$T^8=0$			$\Lambda^8=10$		

Table 1: Parameters for simulation with Gaussian noise.

Figure 3 shows the de-noised signal. Note that our algorithm is able to capture both the initial parts of the signal (consisting of smoother portions with some jumps in between) as well as the later portions with Brownian texture. Thus, the controlled smoothing that our algorithm gives enables it to perform better than any conventional filtering operation. The SNR improvement from 6db to 12.9db is somewhat better than that obtained by the de-noising algorithm of [5]-[6], ([5] obtains improvement from 6db to 12.1db); but more importantly the reconstruction follows the transients in the noiseless signal better than in [5], especially in the portion with Brownian texture. Figure 4 shows the signal of Figure 1 with added noise consisting of segments of different distributions- uniform (samples 1 to 50), gamma (with gamma-parameter 3, samples 51 to 100), exponential (samples 101 to 180) and Gaussian (samples 181

to 256). The algorithm was run with the Daubechies 6-tap wavelet filter, and the parameters t^j and λ^j as shown in Table 2.

j	1	2	3	4	5	6	7	8
t^j	2	2	5	2	2	1.2	0	0
λ^j	0.00001	0.0001	2	30	1000	1000	10	10
$J=8$			$T^8=0$			$\Lambda^8=10$		

Table 2: Parameters for simulation with non-stationary noise.

Figure 5 shows the output of the algorithm in this case. Again we see that both the smoother parts as well as the later irregular portions of the input signal have been captured by the algorithm.

5. CONCLUSION

We conclude that filtering in the wavelet domain, using smoothing with regularization and with the gradient-based switches, is able to recover both smooth and irregular parts of the input signal from the noisy signal. Moreover the algorithm is able to handle non-stationary or variable distribution noise.

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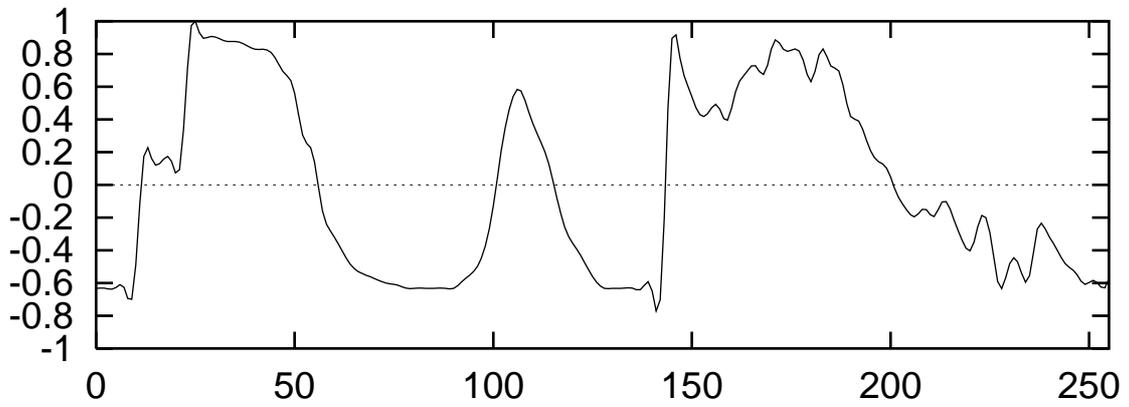


Figure 1. Input signal, free of noise.

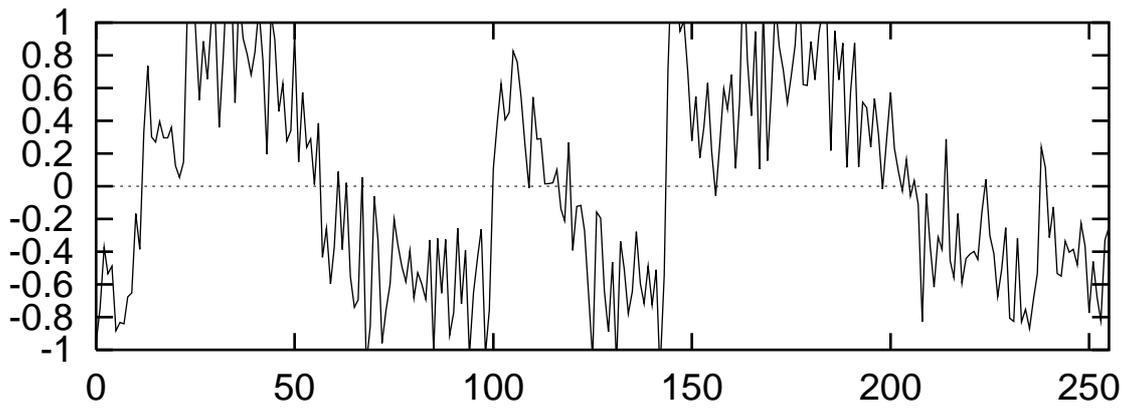


Figure 2. Input signal with Gaussian noise added, SNR=6db.

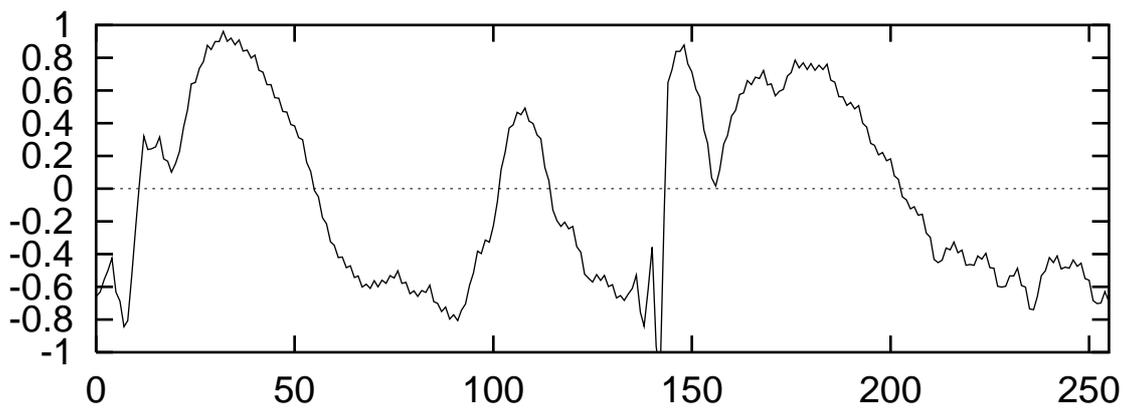


Figure 3. De-noised signal obtained from our algorithm, SNR=12.9db.

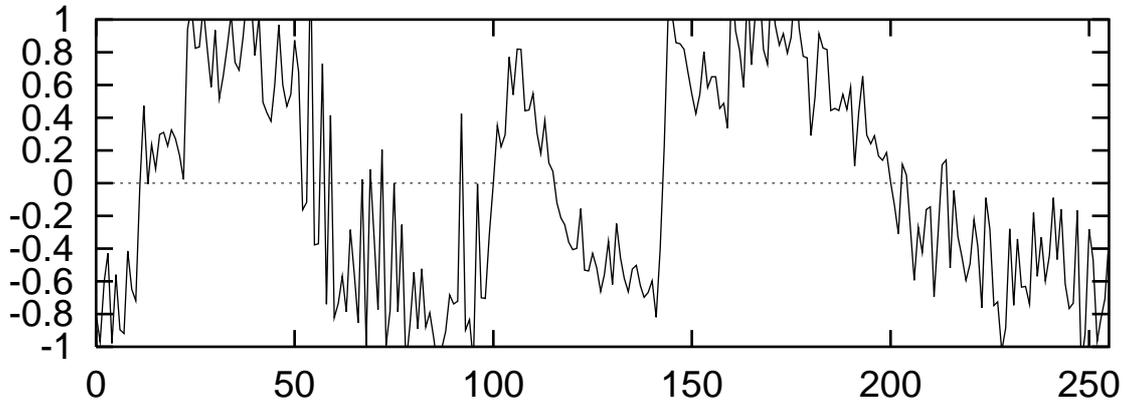


Figure 4. Input signal with non-stationary noise added, SNR=6db.

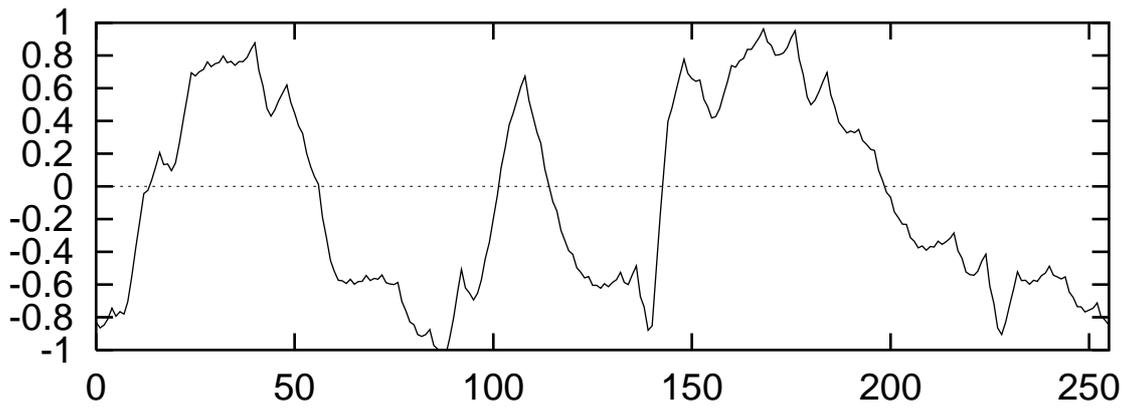


Figure 5. De-noised signal obtained from our algorithm, SNR=10.5db.