EXACT MULTICHANNEL DECONVOLUTION ON RADIAL DOMAINS

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ABSTRACT

A novel multisensor approach to deconvolution is developed. This theory circumvents the ill-posedness inherent in convolution equations by overdetermining the input signal by a multichannel system of convolvers $\{\mu_i\}$, chosen so that any information lost by one channel is retained by another. The deconvolution problem is then solved by constructing "deconvolvers" that allow us to construct the Dirac δ by filtering each μ_i by its deconvolver, and then adding the filtered channels together. This in turn allows us to reconstruct the original signal f. The process is linear and stable with respect to noise. The general multichannel theory is discussed. The deconvolution theory in radially symmetric domains is then developed in greater detail.

1. INTRODUCTION

Linear, translation invariant systems (e.g., sensors, linear filters) are modeled by the convolution equation $s = f * \mu$, where f is the input signal, μ is the system impulse response function (or, more generally, impulse response distribution), and s is the output signal. We refer to μ as a convolver. In many applications, the output s is an inadequate approximation of f, which motivates solving the convolution equation for f, i.e., deconvolving f from μ. If the function μ is time-limited (compactly supported) and non-singular, we have shown that this deconvolution problem is ill-posed in the sense of Hadamard (see [7]).

A theory of solving such equations has been developed. It circumvents ill-posedness by using a multichannel system. If we overdetermine the signal f by using a *system* of convolution equations, $s_i = f * \mu_i$, $i = 1, \ldots, n$, the problem of solving for f is well-posed if the set of convolvers $\{\mu_i\}$ satisfies the condition of being what we call *strongly coprime*. In this case, there exist compactly supported distributions (deconvolvers)

ν_i, $i = 1, \ldots, n$ such that

$$\widehat{\mu_1} \cdot \widehat{\nu_1} + \ldots + \widehat{\mu_n} \cdot \widehat{\nu_n} = 1 \, .$$

Transforming, we get

$$\mu_1 * \nu_1 + \ldots + \mu_n * \nu_n = \delta \, ,$$

which in turn gives

$$s_1 * \nu_1 + \ldots + s_n * \nu_n = f \, .$$

We describe the strongly coprime condition, and we give examples of sets of strongly coprime system response functions and their deconvolutions for functions in one and several variables. In the language of applications, the set of convolvers $\{\mu_i\}$ models a linear translation invariant multichannel system consisting of an array of sensors or filters. The system is created so that *no* information contained in the input signal f is lost. The signal f is gathered by this system as $\{s_i = f * \mu_i\}$. The signals s_i are then filtered by the ν_i (which have been created digitally, optically, etc., in coordination with the creation of the system and possibly tailored to be optimized under some constraint) and added, resulting in the reconstruction of f. We discuss the various classes of impulse response functions modeled by the theory. We close by discussing how this deconvolution technique works in radially symmetric domains.

2. GENERAL MULTICHANNEL THEORY

The deconvolution problems we consider are for convolvers μ which are realistic mathematical models of the impulse response functions of linear translation invariant systems. Therefore, we exclude distributions which are not compactly supported (since one would have to integrate for all time to get any information from such a system), distributions of order $k \geq 1$ (since

one would have to impose smoothness conditions on any input functions), and any measures that are singular with respect to Lebesgue measure (since such a system is impossible to build). A distribution μ is a *realizable convolver* if μ is a compactly supported finite Borel measure which is absolutely continuous with respect to Lebesgue measure on $\mathbf{R}^n$. We have shown the following.

Theorem 1 ([7]) *Let $\mu(t)$ be a realizable convolver. Then for $f \in C(\mathbf{R}^n)$, the convolution operator $\mathcal{C}_\mu(f) = f * \mu$ is not injective. If $f \in L^2(\mathbf{R}^n)$, we have that $\mathcal{C}_\mu \colon L^2(\mathbf{R}^n) \longrightarrow \mathcal{R}_\mu \subset L^2(\mathbf{R}^n)$ is injective. However, the deconvolution operator $\mathcal{D}_\mu(f * \mu) = f$ is an unbounded and therefore discontinuous linear operator. Thus, the deconvolution problem of recovering f from $f * \mu$ is ill-posed in the sense of Hadamard.*

We have constructed solutions to the deconvolution problem for various classes of compactly supported convolvers, assuming only that our input or initial functions f are of finite energy. Ill-posedness is circumvented by creating a multichannel system. Each channel in the system is a convolver, and the system overdetermines the input function with all of the convolvers in an array chosen so that any information lost by one convolver is retained by another. This theory of deconvolution has its roots in the work of Wiener [12] and Hörmander [10], and has been developed into a working theory by Berenstein, Taylor, Gay, Yger, et al. [1] – [8]. These methods are both linear (convolution with deconvolvers) and realizable (the support of the deconvolvers being contained in the bounded support of the kernels of the convolution equations). Thus, deconvolution at a point $t \in \mathbf{R}^n$ depends only on data near t. The theory assumes no *a priori* information about the input signals. Moreover, the theory can be used to develop a stable system for complete signal recovery.

The theory starts with the following result of Hörmander [10]. For the compactly supported distributions $\{\mu_i\}_{i=1}^n$ on $\mathbf{R}^n$, there exist compactly supported distributions $\{\nu_i\}_{i=1}^n$ such that $\mu_1 * \nu_1 + \ldots + \mu_n * \nu_n = \delta$ if and only if there exist positive constants A and B and a positive integer N such that

$$\left(\sum_{i=1}^n |\widehat{\mu}_i(\zeta)|^2 \right)^{\frac{1}{2}} \geq A e^{-2\pi B |\Im \zeta|} (1 + |\zeta|)^{-N} \ , \ \zeta \in \mathbf{C}^n \ .$$

A set of convolvers $\{\mu_i\}_{i=1}^n$ that satisfy the inequality in the theorem is said to be *strongly coprime*. We note that the only way a single compactly supported convolver can be strongly coprime is for it to be a translate of the identity convolver; that is, for it to be the Dirac delta or a translation thereof. This result is one way

to state the general ill-posedness of single convolution equations under the constraints imposed by the conditions of the theorem. Now, let $\{F_i\}_{i=1}^n$ be a given set of transforms of compactly supported distributions. A solution to the *analytic Bezout equation*

$$\sum_i F_i(\zeta) G_i(\zeta) = 1$$

is a set $\{G_i\}_{i=1}^n$ of transforms of compactly supported distributions that satisfies the equation. By Paley-Wiener-Schwartz (PWS) and basic properties of the Fourier-Laplace transform, a solution to the analytic Bezout equation is equivalent to solving for a set of compactly supported distributions $\{\nu_i\}$ such that $\sum_i \mu_i * \nu_i = \delta$, for a given set $\{\mu_i\}$ of compactly supported distributions. By Hörmander's theorem, a strongly coprime set $\{\mu_i\}$ is precisely a set for which the analytic Bezout equation has a solution. The strongly coprime condition guarantees not only that the transforms of the convolvers have no common zeros, but also that these zeros do not cluster too quickly as $|\zeta| \to \infty$. Thus, if a given signal f is overdetermined by a strongly coprime system of convolution equations, $s_i = f * \mu_i$, $i = 1, \ldots, n$, then the problem of solving for f is well-posed. We solve for a set $\{\widehat{\nu}_i(\zeta)\}$ of deconvolvers which satisfy the analytic Bezout equation

$$\sum_i \widehat{\mu}_i \widehat{\nu}_i = 1 \ .$$

Taking inverse transforms of both sides of Bezout gives

$$\sum_i \mu_i * \nu_i = \delta \ .$$

The fact that the strongly coprime condition is the inversion of the PWS growth bound allows us to solve for deconvolvers that are compactly supported. Thus, the deconvolution problem can be solved by constructing the Dirac δ for a given class of convolvers. To construct compactly supported deconvolving *functions*, we begin by solving a more general analytic Bezout equation, i.e., for given analytic $\widehat{\mu}_i$ and $\widehat{\psi}$ satisfying PWS growth conditions, solving for analytic $\widehat{\nu}_i$ satisfying PWS growth conditions such that

$$\widehat{\mu_1} \cdot \widehat{\nu_1} + \ldots + \widehat{\mu_n} \cdot \widehat{\nu_n} = \widehat{\psi} \ .$$

Moreover, we want $\widehat{\psi} = \widehat{\psi_\lambda}$, with $\widehat{\psi_\lambda} \to 1$ as $\lambda \to \infty$ ($\widehat{\psi_\lambda}$ is the transform of an approximate identity). This gives us deconvolving functions, i.e., deconvolvers $\{\nu_{i,\psi}\}$ such that $\mu_1 * \nu_{1,\psi} + \ldots + \mu_n * \nu_{n,\psi} = \psi$, which in turn give

$$(f * \mu_1) * \nu_{1,\psi} + \ldots + (f * \mu_n) * \nu_{n,\psi} = f * \psi = f_\psi \ .$$

Then, as $\psi \to \delta$, $f_\psi \to f$ in the sense of distributions. The deconvolvers in these implementable formulae are periodic functions expressed in their Fourier series expansions (see [7]).

We give the following specific example. Let $\mu_1(t) = \chi_{[-1,1]}(t)$, $\mu_2(t) = \chi_{[-\sqrt{p},\sqrt{p}]}(t)$ model the impulse response of the channels of a two–channel system. An examination of the Fourier–Laplace transforms $\widehat{\mu_i}(\zeta)$, $i = 1, 2$ gives that $\{\mu_i\}$ are strongly coprime (see [4]). We develop compactly supported deconvolvers which construct an arbitrarily close approximation ψ of the Dirac δ ([7]). The formulae for the $\{\nu_i\}$ relative ψ are expressed as a Fourier series. (The advantage of constructing ψ instead of the δ is that we can express our formulae for the deconvolvers as functions, and not as distributions.) The smoothness and the size of the support of the auxiliary function ψ guarantee that the deconvolvers are compactly supported. For the compactly supported function ψ, we have that $|\widehat{\psi}(z)| \le \frac{d}{(1+|z|)^{3+\eta}}$ for $z \in \mathcal{Z}_1 \cup \mathcal{Z}_2$ if ψ is in the Hölder space $C^{3+\eta}$. This is sufficient to guarantee that the series representations for the ν_i converge to compactly supported functions.

Theorem 2 ([7]) *The set $\mu_1(t) = \chi_{[-1,1]}(t)$, $\mu_2(t) = \chi_{[-\sqrt{p},\sqrt{p}]}(t)$ is a strongly coprime pair of convolvers. Let $f \in L^2(\mathbf{R})$, and for $\eta > 0$ let ψ be a function with support in $[-(1 + \sqrt{p}), (1 + \sqrt{p})]$ in the Hölder space $C^{3+\eta}$ such that $\psi \ge 0$ and $\int_{\mathbf{R}} \psi(t)\, dt = 1$. The deconvolvers $\nu_{i,\psi}$ such that $f * \psi = (f * \mu_1) * \nu_{1,\psi} + (f * \mu_2) * \nu_{2,\psi}$ are given by the formulae*

$$\nu_{1,\psi}(t) = \Big(\frac{1}{2\sqrt{p}} \sum_{j \ne 0} (-1)^{j+1} \frac{\overline{\widehat{\psi}(j/2\sqrt{p})}}{\overline{\widehat{\mu_1}(j/2\sqrt{p})}} +$$

$$\frac{1}{2\sqrt{p}} \sum_{n \ne 0} \frac{\overline{\widehat{\psi}(n/2\sqrt{p})}}{\overline{\widehat{\mu_1}(n/2\sqrt{p})}} e^{\pi i (n/\sqrt{p}) t}\Big) \times \chi_{[-\sqrt{p},\sqrt{p}]},$$

$$\nu_{2,\psi}(t) = \Big(\frac{1}{2} \sum_{j \ne 0} (-1)^{j+1} \frac{\overline{\widehat{\psi}(j/2)}}{\overline{\widehat{\mu_2}(j/2)}} +$$

$$\frac{1}{2} \sum_{n \ne 0} \frac{\overline{\widehat{\psi}(n/2)}}{\overline{\widehat{\mu_2}(n/2)}} e^{\pi i n t}\Big) \times \chi_{[-1,1]}.$$

The function $f * \psi$ is an arbitrarily close approximation of f which converges to f in the sense of distributions as $supp(\psi) \to \{0\}$.

This development works for other classes of convolvers and filters. The current stock of convolvers and their associated deconvolvers includes characteristic functions of squares, (hyper)cubes (see [2], [6], and [7]) and disks and n–dimensional balls (see [3], [4]).

The theory has been expanded in one variable to more general convolvers, including convolvers modeled by linear combinations of characteristic functions, linear combinations of n–fold convolutions of characteristic functions with equally space knots (cardinal splines), and truncated sinc, cosine, and Gaussian functions. We have shown the conditions for a strongly coprime set of convolvers $\{\mu_i\}$ for each of these types of functions, which, as in the example above, is an arithmetic condition on the zero sets of the Fourier–Laplace transforms $\{\widehat{\mu_i}\}$. We then have solved for deconvolvers $\{\nu_{i,\psi}\}$ such that $\psi = \mu_1 * \nu_{1,\psi} + \ldots + \mu_n * \nu_{n,\psi}$, where ψ is an approximate convolution identity, by solving the modified Bezout equation $\widehat{\psi} = \widehat{\mu_1} \cdot \widehat{\nu_{1,\psi}} + \ldots + \widehat{\mu_n} \cdot \widehat{\nu_{n,\psi}}$.

In several variables, the formulae for the deconvolvers is simplified by not only solving for an approximation to the δ, but also by strengthening the strongly coprime condition [2]. This paper gives the solution to the deconvolution problem when the convolvers are (hyper)rectangular regions in $\mathbf{R}^n$.

3. DECONVOLUTION ON RADIAL DOMAINS

The Fourier transform on radially symmetric domains is expressed in terms of Bessel functions, and is generally referred to as the Fourier-Bessel transform [9]. As one might expect, conditions for exact deconvolution on these domains is also expressed in terms Bessel functions; in particular, in terms of their zeros.

Deconvolution on circularly symmetric domains can be expressed as a *Pompeiu* problem. Let χ_1, χ_2 be the characteristic functions of the disks $B(0, r_1)$, $B(0, r_2)$, and let E be the collection of positive quotients of zeros of the Bessel function J_1.

Theorem 3 ([3]) *Let $\xi, \eta \in E$ with $\xi, \eta > 0$. If there exists an $\alpha > 0$ such that*

$$\Big|\frac{r_1}{r_2} - \frac{\xi}{\eta}\Big| \ge \frac{1}{\alpha}\eta^{-\alpha},$$

then the mapping $P : f \to (\chi_1 * f + \chi_2 * f)$ is injective, and moreover there exists ν_1, ν_2 such that

$$f = \nu_1 * (\chi_1 * f) + \nu_2 * (\chi_2 * f).$$

We say that $\frac{r_1}{r_2}$ is poorly approximated by elements of E.

In [3], this theorem was also extended to the local problem of reconstructing f in some disk $B(0, R)$, $R > r_1 + r_2$, from its averages on $B(0, r_1)$, $B(0, r_2)$.

This result fits in the larger context of Pompeiu problems. Let $E_1, \ldots, E_m$ be compact sets of positive measure in $\mathbf{R}^n$, let $C(\mathbf{R}^n)$ denote the space of

continuous functions, and let $M(n)$ be the group of Euclidean motions in $\mathbf{R}^n$. Then, the (global) Pompeiu transform associated to the sets $E_1, \ldots, E_n$ is the mapping $P : C(\mathbf{R}^n) \longrightarrow C(M(n))^m$ given by

$$(Pf)(g) = \left(\int_{gE_1} f \, dx, \ldots, \int_{gE_m} f \, dx \right).$$

The problem is then to give conditions on the sets $E_1, \ldots, E_m$ to guarantee that the mapping P is injective. If P is injective, we say that $E_1, \ldots, E_m$ have the *Pompeiu property*. We can construct the inverse mapping by constructing deconvolvers which recover the function f, as in the theorem above.

A single disk or ball never has the Pompeiu property. However, a pair that satisfies certain conditions does.

Theorem 4 ([3]) *Let*

$$Z_n = \{ \frac{\xi}{\eta} : \xi, \eta > 0, J_{n/2}(\xi) = J_{n/2}(\eta) = 0 \} .$$

A pair of balls $B(0, r_1)$, $B(0, r_2)$, has the Pompeiu property if and only if $\frac{r_1}{r_2} \notin Z_n$.

Thus, for such a pair, deconvolution is possible, for the inverse mapping is created by constructing deconvolvers.

We can also formulate a local formulation of this, by restricting ourselves to open sets U that can be covered by balls of a fixed radius R.

Theorem 5 ([3]) *Let $r_1, r_2 > 0$, $\frac{r_1}{r_2} \notin Z_n$, and $R > r_1 + r_2$. Then the pair of balls $B(0, r_1)$, $B(0, r_2)$, has the local Pompeiu property with respect to $B(0, R)$. Under the additional condition that $\frac{r_1}{r_2}$ is poorly approximated by elements of Z_n, the condition $R > r_1 + r_2$ is also necessary.*

The deconvolvers for these theorems are expressed in terms of Fourier-Bessel series, much in the same manner as in Theorem 2. We are currently looking to develop approximate deconvolvers in B-splines.

4. REFERENCES

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