SUFFICIENT STABILITY BOUNDS FOR SLOWLY VARYING DISCRETE-TIME RECURSIVE LINEAR FILTERS

Alberto Carini[1] *V.John Mathews*[2] *Giovanni L. Sicuranza*[1] *

[1] DEEI, Università degli Studi di Trieste, Via Valerio 10, 34127 Trieste, Italy
[2] Department of Electrical Engineering, University of Utah, Salt Lake City, UT 84112, USA

ABSTRACT

This paper derives sufficient time-varying bounds on the maximum variation of the coefficients of an exponentially stable, linear, time-varying and recursive filter. The stability bound is less conservative than all previously derived bounds for time-varying IIR systems. The bound is then applied to control the step size of output error adaptive IIR filters to achieve exponentially stable operation. Experimental results that demonstrate the good stability characteristics of the resulting algorithms are included in this paper.

1. INTRODUCTION

Adaptive IIR filters have been the subject of active research over the last three decades [3], [4], [5], [6], [9]. Despite a large amount of work that has been done, some open issues still remain. One of these issues is that of ensuring the stability of the time-varying IIR filter that results from the adaptation process.

Researchers have attempted to derive adaptive IIR filters that operate in a stable manner in several different ways. One class of algorithms that includes the hyperstable adaptive recursive filter (HARF) [4] requires a certain system transfer function to be strictly positive real (SPR), which is not easy to guarantee in practice. Another class of algorithms employ lattice structures [6]. Such filters are guaranteed to be stable if the reflection coefficients that are computed adaptively are bounded by one. However, there are many applications in which direct form coefficients are required, and conversion from lattice to direct form is not computationally efficient. A third class of adaptive IIR filters employ stability monitoring by checking the location of the instantaneous poles of the system and projecting the coefficients back to a region for which the instantaneous poles are within the unit circle [4]. Unfortunately, time-varying filters may be unstable even when the instantaneous poles are within the unit circle. Consequently, even though projection-based techniques work well in a large number of situations, they are not guaranteed to operate in a stable manner for all input signals.

This paper presents a method for controlling the adaptation step size to guarantee exponential stability of output error adaptive IIR filters. This method is based on a novel, sufficient time-varying bound on the maximum allowable coefficient variation of exponentially stable time-varying linear recursive filters with instantaneous poles inside the unit circle. This bound is less conservative than all previously derived bounds that have been found in the literature [1], [2]. It is well-known [1], [2], [6], [8] that if the poles of a recursive time-varying linear system are always inside the unit circle and if they are sufficiently slowly-varying, then the recursive system itself is exponentially stable. Desöer was the first to prove this result for discrete-time systems and to determine a sufficient upper bound for the maximum variation of the entries of the system matrix [2]. An additional improvement on this bound was recently presented in [1]. In principle, we can develop exponentially-stable adaptive IIR filters by constraining their instantaneous poles to be inside the unit circle and by using a sufficiently small step size that forces the coefficient variations to be bounded by the sufficient bound described in [1] or [2]. However, such stability bounds are very restrictive in the sense that the speed of convergence of the adaptive filters that limit the step size sequence by these sufficient conditions is too slow to be of use in many practical applications. The upper bound we derive for the maximum allowable variation of the filter coefficients is much less restrictive.

The rest of this paper is organized as follows. The new stability bound is derived in Section 2. This bound is then transformed into a bound on the step size of the adaptive filter. The maximum step size and other stability conditions obtained in Section 2 are applied to ensure the exponential stability of an adaptive output error algorithm in Section 3. Simulation results presented in this section confirm the good stability properties of the stabilized adaptive algorithm.

2. SUFFICIENT STABILITY CONDITIONS FOR SLOWLY VARYING DISCRETE-TIME, LINEAR, RECURSIVE SYSTEMS

We consider a linear, time-varying and recursive system with input-output relationship given by

$$y(k) = \sum_{i=0}^{N-1} b_i(k)u(k-i) + \sum_{i=1}^{N-1} a_i(k)y(k-i). \tag{1}$$

Let

$$\theta(k) = [b_0(k), \ldots, b_{N-1}(k), a_1(k), \ldots, a_{N-1}(k)]^T \tag{2}$$

denote the coefficient vector and let the evolution of the coefficients be of the form

$$\theta(k+1) = \theta(k) + \mu_k \psi(k), \tag{3}$$

where μ_k is a time-varying scalar sequence. Our objective is to find a sufficient bound on the squared-norm of the increment vector $\mu_k\psi(k)$ given by $\mu_k^2\psi^T(k)\psi(k)$ such that the time-varying system of (1) is exponentially stable. From

*This Work was partially supported by NATO Grant CRG. 950379 and Esprit LTR Project 20229 Noblesse

such a result, we can immediately find a bound on μ_k for guaranteeing the stability of the system. An adaptive filter with coefficient update as in (3) will be exponentially stable if μ_k is chosen smaller than or equal to such a bound. The basis for our work is the following theorem proved in [8]:

Theorem 1 *The linear state equation*

$$X(k+1) = A(k)X(k), \qquad X(k_0) = X_0 \tag{4}$$

is uniformly exponentially stable if and only if there exists an $N \times N$ matrix sequence $Q(k)$ that is symmetric for all k and such that

$$\eta I \leq Q(k) \leq \rho I \tag{5}$$

and

$$A^T(k+1)Q(k+1)A(k+1) - Q(k) \leq -\phi I, \tag{6}$$

where η, ρ and ϕ are finite positive constants.

In the theorem, the condition "matrix $Q \leq \rho I$" implies that $x^TQx \leq \rho x^Tx$ for all vectors x.

One candidate sequence for $Q(k)$ is the unique, symmetric and positive definite solution of the discrete-time Lyapunov equation

$$A^T(k)Q(k)A(k) - Q(k) = -I_N. \tag{7}$$

The solution for $Q(k)$ is [8]

$$\text{vec}[Q(k)] = -[A^T(k) \otimes A^T(k) - I_{N^2}]^{-1}\text{vec}[I_N], \tag{8}$$

where vec$[Q(k)]$ is a vector formed by stacking all the columns of $Q(k)$ and $\otimes$ denotes the Kronecker product. This choice of the matrix $Q(k)$ satisfies the bounds on the spectral norm in the theorem if the instantaneous poles of the system are inside the unit circle at all times [8]. It is straightforward to show using (7) that

$$A^T(k+1)Q(k+1)A(k+1) - Q(k) = Q(k+1) - Q(k) - I_N. \tag{9}$$

The problem of deriving a sufficient stability bound boils down to finding the maximum allowable coefficient variations such that

$$\|Q(k+1) - Q(k)\| \leq 1, \tag{10}$$

where $\|(\cdot)\|$ denotes the spectral norm of the matrix $(\cdot)$. Given the above result, we can check if a recursive linear system is stable in two steps:

1. Verify that the instantaneous poles of the system are inside the unit circle.
2. Determine if condition (10) is met for our choice of the step size μ_k .

Using (10), we can determine an upper bound for the step size μ_k in (3) that guarantees the stability of the recursive system described in (1)–(3). First, the following inequality holds:

$$\|Q(k+1) - Q(k)\| \leq \|\text{vec}\big[Q(k+1)\big] - \text{vec}\big[Q(k)\big]\|. \tag{11}$$

In the hypothesis of slowly varying coefficients, the following approximation can be applied:

$$\text{vec}\big[Q(k+1)\big] - \text{vec}\big[Q(k)\big] \simeq \nabla_\theta \text{vec}\big[Q(k)\big] \cdot \Delta\theta(k), \tag{12}$$

where ∇_θ indicates the gradient vector operator with respect to the coefficient vector θ and $\Delta\theta(k) = \theta(k+1) - \theta(k)$. We note from (3) that

$$\Delta\theta(k) = \mu_k \psi(k). \tag{13}$$

From (10) and (11) we can derive a sufficient condition for the exponential stability of the system in (4) to be

$$\|\text{vec}\big[Q(k+1)\big] - \text{vec}\big[Q(k)\big]\| \leq \zeta < 1. \tag{14}$$

We can substitute the approximation of (12) in the above condition and manipulate the resulting expression to obtain an explicit condition on μ_k for the stability of (4) to be

$$\mu_k < \frac{\zeta}{\|\nabla_\theta \text{vec}\big[Q(k)\big] \cdot \psi(k)\|}. \tag{15}$$

The stability condition of (10) is derived without resorting to any approximation. However, this condition can be employed in adaptive recursive filtering applications only with the help of projection techniques. Even though the derivation of (15) employs an approximation that is based on slow variations in the coefficients, this condition has the advantage of being useful in directly controlling the step size of adaptation. In all the experiments we conducted, the stability conditions of (10) and (15) gave similar results. Moreover, this stability bound is less conservative than the bounds available in [1] and [2] for time-varying recursive linear systems.

2.1. The Second-Order Case

The stability condition derived in the previous subsection can be easily converted into explicit expressions in the filter coefficients for time-varying second-order systems. Even though the conditions we derived hold for any filter order, second-order systems are of particular importance because the implementation of the stability conditions is simplest when the adaptive filter is realized as a cascade or parallel connection of second-order sections. We consider the following second-order filter:

$$\begin{aligned} y(k) \quad = \quad & a_1(k)y(k-1) + a_2(k)y(k-2) + b_0(k)u(k) + \\ & b_1(k)u(k-1) + b_2(k)u(k-2). \end{aligned} \tag{16}$$

It is easy to check if the instantaneous poles of a second-order system are inside the unit circle. The coefficients of the filter $a_1(k)$ and $a_2(k)$ must satisfy the inequalities

$$|a_1(k)| + a_2(k) < 1 \tag{17}$$

and

$$a_2(k) > -1. \tag{18}$$

for all the instantaneous poles to be bounded by one.

The candidate Lyapunov matrix $Q(k)$ for the system of (16) is given by

$$Q(k) = \begin{bmatrix} -2\frac{a_2(k)-1}{r(k)} & 2\frac{a_1(k)a_2(k)}{r(k)} \\ 2\frac{a_1(k)a_2(k)}{r(k)} & -\frac{s(k)}{r(k)} \end{bmatrix}, \tag{19}$$

where $r(k) = -a_2^3(k) + a_2^2(k) + a_1^2(k)a_2(k) + a_2(k) + a_1^2(k) - 1$ and $s(k) = a_2^3(k) - a_2^2(k) + a_1^2(k)a_2(k) + a_2(k) + a_1^2(k) - 1$. Let

$$Q(k+1) - Q(k) = \begin{bmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{bmatrix}. \tag{20}$$

The characteristic polynomial of this matrix is

$$p(\lambda) = \lambda^2 - (d_{11} + d_{22})\lambda + d_{11}d_{22} - d_{12}^2. \quad (21)$$

Verifying the condition of (10) is equivalent to determining if the characteristic roots of (21) are bounded by one. Thus, the stability test verifies if

$$-d_{11}d_{22} + d_{12}^2 > -1 \quad (22)$$

and

$$|d_{11} + d_{22}| - d_{11}d_{22} + d_{12}^2 < 1. \quad (23)$$

The condition in (17) and (18), and the condition in (22) and (23) can be applied using a projection technique to ensure the exponential stability of the time-varying and recursive, second-order filter. Since the number of iterations necessary to meet these conditions is not known *a priori*, it is preferable to limit the step size μ_k by means of condition (15). This condition translates to the following bound on the step size for a second-order system:

$$\mu_k \leq \frac{r^2(k)}{\sqrt{\begin{array}{c} 4\big(r(k)\cdot\psi_2(k) - (a_2(k)-1)\cdot v(k)\big)^2 + \\ 8\big((a_1(k)\psi_2(k) + a_2(k)\psi_1(k))\cdot r(k) + \\ -a_1(k)a_2(k)\cdot v(k)\big)^2 + \big(r(k)\cdot w(k) - v(k)\cdot s(k)\big)^2 \end{array}}}, \quad (24)$$

where $w(k) = \big(3a_2^2(k) + a_1^2(k) - 2a_2(k) + 1\big)\psi_2(k) + 2a_1(k)\big(a_2(k)+1\big)\psi_1(k)$ and $v(k) = \big(-3a_2^2(k) + 2a_2(k) + a_1^2(k) + 1\big)\psi_2(k) + 2a_1(k)\big(a_2(k)+1\big)\psi_1(k)$.

We point out again that it is necessary to check if the instantaneous poles of the updated filter are inside the unit circle at each time in order to ensure stable operation of the system.

The computational complexity of calculating the step size bound in (24) correspond to 16 multiplications, one square-root operation and one division per second-order section. Consequently, the complexity of implementing the stability bounds for a cascade or parallel adaptive filter is linearly proportional to the order of the filter. Furthermore, this complexity is comparable to or smaller than the complexity of adapting the coefficients in many adaptive IIR filtering algorithms.

3. EXPERIMENTAL RESULTS

We now present the results of experiments that demonstrate the usefulness of the bounds and the adaptive filters that utilize these bounds. The experiments also compare the performance of the stabilized adaptive filters with the SHARF algorithm.

In the first set of results presented below, the adaptive filters were employed to identify an unknown, fourth-order IIR filter with transfer function

$$H(z) = \frac{1}{1 - 1.86z^{-1} + 0.8698z^{-2}} + \frac{2}{1 - z^{-1} + 0.5z^{-2}} \quad (25)$$

using measurements of the input and output signals. The poles of the unknown system are located at $[0.93 \pm 0.07j]$ and $[0.5 \pm 0.5j]$. The adaptive filters employed a parallel connection of two second-order systems and were adapted using an appropriate variation of the Gauss-Newton algorithm [4]. The input of the unknown system was a colored Gaussian signal obtained by filtering a white Gaussian signal with zero mean value and unit variance with an FIR filter whose transfer function was given by

$$W(z) = 1 + 0.5z^{-1}. \quad (26)$$

The desired response signal was generated by processing this signal with the unknown system and then corrupting the output with an additive, zero-mean and white Gaussian noise sequence that is statistically independent of the input signal. The variance of the measurement noise was such that the output signal-to-noise ratio was 30 dB. The adaptive filter employed a different step size sequence for each second-order section and for the recursive and non-recursive part of each section. The step size of the recursive part was selected to be the minimum of 0.001 or the bound suggested by our conditions, while that of the moving average part was 0.0005. The forgetting factor used in the evaluation of the inverse of the autocorrelation matrix was 0.9999. Almost all output error adaptive recursive filters are susceptible to converging to the wrong local minima of the squared estimation error surface. In the results presented here, we compare the speed of convergence of various algorithms to the correct solution. In order to do so, all the experiments that resulted in convergence to wrong local minima were eliminated from the calculation of the ensemble averages. The results displayed in the figures are averages of the first fifty experiments in which the coefficients converged to the correct solution.

Figures 1a and 1d display the evolution of the ensemble averages of the coefficients of the parallel section that corresponds to the coefficients 1.86 and 0.8698 of the unknown system for two different realizations of the stabilized adaptive filter. Figures 1b and 1e provide similar comparisons for the step size, while Figures 1c and 1f compare the ensemble averaged squared estimation errors. Figures 1a-1c display the results obtained by implementing the output error algorithm with the stability condition of (24). Figures 1d-1f present the corresponding results obtained using a projection algorithm employing the conditions in (22)-(23). We can see from these results that the approximations assuming slow-variations of the coefficients to obtain (24) provide reliable step size bounds. It appears that the approximations resulted in more conservative bounds than the the conditions in (22)-(23), resulting in slightly slower convergence speed. The initial values of the step size are small in this example because the initial estimation error values are large. Combined with the large error, the initial values of the step size produced the largest changes possible that still maintained the exponential stability of the system.

We now compare the performance of the stabilized adaptive IIR filter using the closed form conditions in (24) with that of the SHARF algorithm. In order to make the comparisons as fair as possible, we used a single second order system for identifying an unknown second-order system with transfer function

$$H(s) = \frac{1}{1 - 1.9z^{-1} + 0.905z^{-2}}. \quad (27)$$

We used the same experimental conditions as in the previous example, with the difference that we employed the same step size sequence for adapting the moving average and the recursive coefficients of the system.

The step size μ was selected to be 0.0001 for the SHARF algorithm so that the steady-state excess mean-square error was identical to that of the stabilized adaptive IIR filter of this paper when the coefficient sequence was selected as in the previous set of experiments.

Figure 2 displays the evolution of the mean-squared estimation error for the two algorithms. We can see from this figure that the SHARF algorithm converges much slower

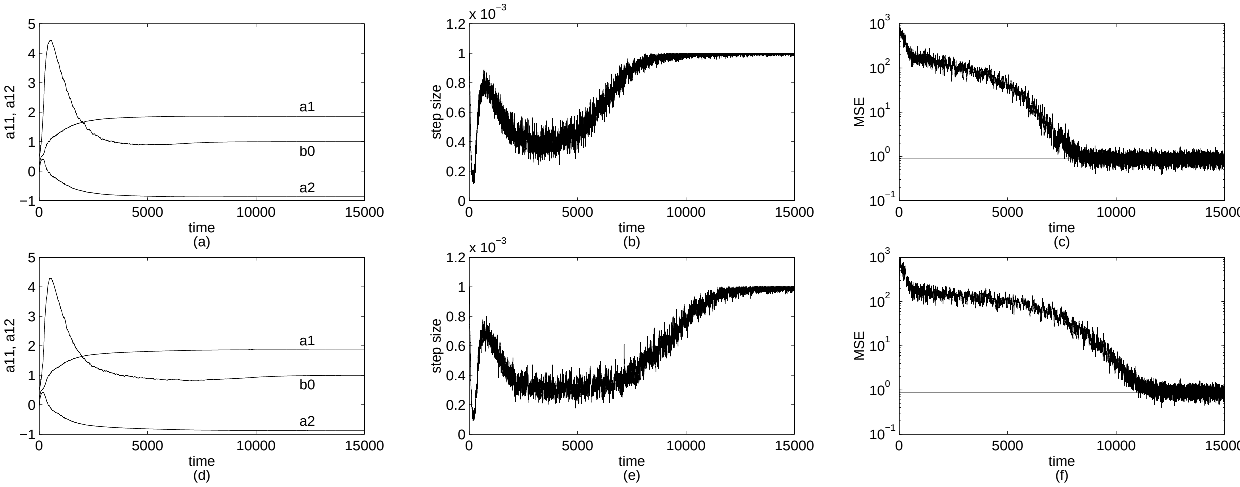

Figure 1. Learning curves for two stabilized adaptive IIR filters.

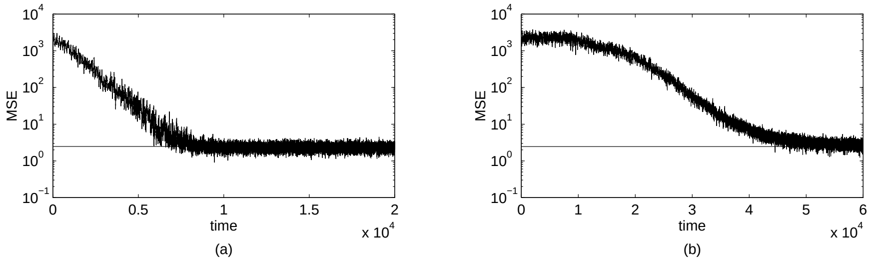

Figure 2. Comparison of the stabilized adaptive filter of this paper and the SHARF algorithm.

than the method introduced in this paper in this experiment.

4. CONCLUDING REMARKS

This paper presented a novel stability condition for time-varying recursive linear systems. The stability bound obtained in (15) is less conservative than all previously derived bounds [1], [2] for time-varying recursive linear systems. In particular, all the bounds converge to zero as the instantaneous poles approach the unit circle. The rate at which $\mu(k)$ approaches zero when the poles tends to the unit circle is several orders of magnitude slower than the bound derived in [2]. The time-varying bound on the step size of (15) may be incorporated into any practical adaptive IIR filter. It is well-known that certain adaptive IIR filtering algorithms such as Feintuch's method diverge for all choices of the step size for certain input signals. Experimental results as well as theoretical considerations indicate that the step size bound derived in this paper eventually goes to zero in such situations, thus preserving the exponential stability of the adaptive filter.

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