# AN EFFICIENT HAAR WAVELET-BASED APPROACH FOR THE HARMONIC RETRIEVAL PROBLEM

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## ABSTRACT

Modern subspace-based algorithms can offer high- resolution spectral estimates but with a cost of high computational complexity for the eigenvalue decomposition (EVD) involved. In this paper, we propose a novel preprocessing scheme which can be used in conjunction with the subspace-based algorithms to alleviate the high computations previously required. The new scheme is to demodulate the input data first, and then takes the computationally efficient discrete-time Haar wavelet transform (HWT). Only the principle subband component (PSC) of the transformed data is kept for further processing, which not only retains the same amount of information but also possesses the same characteristic as that of the original (noiseless) harmonic data. The subspace-based algorithms are thus applicable to this new set of transformed data but with substantially reduced computational load. Some simulation results are provided to justify the proposed approach.

## 1. INTRODUCTION

The harmonic retrieval problem which arises in various areas such as geophysics and radar has been an active research area during the past few decades. Recently, the subspace-based approaches such as the MUltiple SIgnal Classification (MUSIC) [1] and Toeplitz Approximation Method (TAM) [2] (or, equivalently, EStimation of Parameters via Rotational Invariance Techniques (ESPRIT) [3] in this problem [4]) have received considerable amount of attention. Owing to the fact that the underlying model assumed is just a summation of harmonics, these subspace-based methods in general yield superior performance when compared with the traditional Fourier-based algorithms or parametric modeling approaches [5]. The TAM and ESPRIT are, in particular, computationally attractive since they do not need to search over the entire spectral band to locate the desired harmonics.

However, all of these subspace-based methods call for lots of computations since they all rely on the computationally intensive EVD. In this paper, we address a novel discrete-time Haar wavelet-based preprocessing scheme which can be used in conjunction with the existing subspace-based methods to alleviate the computational overhead that would have been required. Some related works using the wavelet transform (or subband decomposition) [6, 7] for the spectral estimation problem have been reported recently. For example, in [8], a subband decomposition FFT procedure is addressed. It, however, suffers the same resolution limitation of the Fourier-based algorithms. [9] considers a wavelet packet-based approach for this problem, in which the Daubechies' filters has been used. However, both of the algorithms of [8, 9] did not fully exploit or preserve the characteristic of the harmonic data which underlines the modern high resolution, low complexity subspace-based algorithms such as [2, 3] for the harmonic retrieval problem.

The proposed approach begins with a demodulation of the input data. After that, unlike [9], a subband decomposition of the demodulated data via the HWT is carried out, which is computationally simpler since only additions and subtractions (except the scaling factor) are involved. Then, only the PSC of the HWT-data is kept for further processing. This new set of transformed data not only retains the same information as that of the original data (assume that the demodulation frequency and the levels of decomposed stages have been appropriately chosen), but it also exhibits, as justified analytically, the "frequency shifting" property observed in [2, 3]. As a consequence, the computationally efficient subspace rotational invariance (SRI) [2, 3] technique employed by the TAM and ESPRIT is also applicable here. Meanwhile, the amount of data need to be processed is reduced, thus leading to substantially computational savings. The provided simulations confirm this new approach.

## 2. A HAAR WAVELET-BASED APPROACH TO HARMONIC RETRIEVAL

### 2.1. Background Review

Consider a set of data $\{x[n]\}_{n=0}^{N-1}$ which contain $d$ sinusoids as

$$x[n] = \sum_{i=1}^{d} |a_i| e^{j(\omega_i n + \phi_i)} = \mathbf{h}^T \mathbf{F}^n \mathbf{t} \qquad (1)$$

where $|a_i|$, $\phi_i$, and $\omega_i$, $i = 1, \ldots, d$, denote, respectively, the amplitude, phase, and (angular) frequency of these $d$ sinusoids, $\mathbf{h}^T = [a_1, \cdots, a_d]$ with $a_i = |a_i| e^{j\phi_i}$, $\mathbf{F} = diag[e^{j\omega_1}, \cdots, e^{jw_d}]$, $\mathbf{t}$ is a $d \times 1$ vector whose elements are all equal to 1, and the superscript $^T$ denotes matrix transposition. Our objective is to extract the frequencies, $\{\omega_i\}_{i=1}^{d}$, out of the observed noisy data $\{\tilde{x}[n]\}_{n=0}^{N-1}$, where $\tilde{x}[n] = x[n] + w[n]$ and $\{w[n]\}$ is the contaminated white noise.

The wavelet transform has recently received considerable amount of attention [6]. This new type of transform can provide octave subband decomposition (multiresolution analysis) of the input data. The simplest one is a two-band

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wavelet transform which just decomposes the input into low and high spectral bands. Various attempts have been made to determine the corresponding (orthogonal) lowpass filter $\mathbf{d}_0$ and highpass filter $\mathbf{d}_1$, most notable the Daubechies's family of wavelets which can be determined by selecting an appropriate number of vanishing moments [6].

In this paper, we focus on the discrete-time Haar wavelet (has only one vanishing moment) out of two reasons. First, this transform can be efficiently implemented via the Haar filter bank which only requires additions and subtractions. Second, the transformed data also possesses the frequency shifting property as that of the original noiseless harmonic data. Hence, the SRI technique can also be used for the transformed data.

The HWT with $I$ levels of decomposed stages can be expressed in the following matrix notation

$$\mathbf{W} = \begin{bmatrix} \mathbf{D}^{(\frac{N}{2^{I-1}})} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_{N-\frac{N}{2^{I-1}}} \end{bmatrix} \cdots \begin{bmatrix} \mathbf{D}^{(\frac{N}{2})} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_{\frac{N}{2}} \end{bmatrix} \mathbf{D}^{(N)} \tag{2}$$

where

$$\mathbf{D}^{(S)} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{d}_0 & \mathbf{o}_2 & \cdots & \mathbf{o}_2 \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{o}_2 & \mathbf{o}_2 & \cdots & \mathbf{d}_0 \\ -- & -- & -- & -- \\ \mathbf{d}_1 & \mathbf{o}_2 & \cdots & \mathbf{o}_2 \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{o}_2 & \mathbf{o}_2 & \cdots & \mathbf{d}_1 \end{bmatrix} \in \mathcal{R}^{S \times S} \tag{3}$$

with $\mathbf{d}_0 = [1,\ 1]$, $\mathbf{d}_1 = [1,\ -1]$, and $\mathbf{o}_2$ denoting a $1 \times 2$ zero vector.

### 2.2. A HWT-based Approach

In this section, we will try to employ the HWT discussed in the previous section to develop a new procedure for the harmonic retrieval problem. The first step of the proposed approach is to demodulate the noiseless signal $\{x[n]\}$ by a specific frequency $\omega_c$, where $\omega_c$ is assumed to be known and locates in the vicinity of the harmonics of interest. If no such *a priori* information is available, we can simply choose many $\omega_c$'s which are distributed over the whole spectral band and perform the demodulation process *in parallel* for these $\omega_c$'s.

The demodulation process can be described by the following matrix expression

$$\mathbf{y} = \mathbf{P}\mathbf{x} \tag{4}$$

where $\mathbf{y} = [y[0], \cdots, y[N-1]]^T$ is the demodulated data, $\mathbf{P} = diag[1, e^{-j\omega_c}, \cdots, e^{-j(N-1)\omega_c}]$ denotes the demodulation matrix, and $\mathbf{x} = [x[0], \cdots, x[N-1]]^T$.

After the demodulation process, we take the HWT of $\{y[n]\}_{n=0}^{N-1}$ and consider the PSC of the transformed data, which corresponds to the lowest subband portion after the Haar subband decomposition, as shown in Fig. 1. It is ready to justify that this manipulation can be expressed as

$$\mathbf{z} = \mathbf{W}_P \mathbf{y} \tag{5}$$

where $\mathbf{z} = [z[0], \cdots, z[N_d - 1]]^T$ ($N_d = \frac{N}{2^I}$, $I$ is the levels of decomposition) denotes the PSC of the transformed data after the WHT and

$$\mathbf{W}_P = \frac{1}{\sqrt{2^I}} \begin{bmatrix} \mathbf{1}_{2^I} & \mathbf{o}_{2^I} & \cdots & \mathbf{o}_{2^I} \\ \mathbf{o}_{2^I} & \mathbf{1}_{2^I} & \cdots & \mathbf{o}_{2^I} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{o}_{2^I} & \mathbf{o}_{2^I} & \cdots & \mathbf{1}_{2^I} \end{bmatrix} \in \mathcal{R}^{N_d \times N} \tag{6}$$

where $\mathbf{1}_{2^I}$ denotes a $1 \times 2^I$ vector whose elements are all equal to 1, and $\mathbf{o}_{2^I}$ denotes a $1 \times 2^I$ zero vector. Note that $\mathbf{W}_P$ corresponds to the first $N_d$ rows of $\mathbf{W}$, and $rank(\mathbf{W}_P) = N_d$. Using (1), it is straightforward to show that the $m^{th}$ component of $\mathbf{z}$ is

$$\begin{aligned} z[m] &= \frac{1}{\sqrt{2^I}} \mathbf{h}^T (\mathbf{I} + \mathbf{F} e^{-j\omega_c} + \cdots + \mathbf{F}^{(2^I - 1)} e^{-j\omega_c (2^I - 1)}) \\ & \quad (\mathbf{F} e^{-j\omega_c})^{2^I m}\ \mathbf{t}, \qquad m = 0, 1, \cdots, N_d - 1 \end{aligned} \tag{7}$$

Assume that $I$ has been appropriately chosen so that $\mathbf{z}$ contains the same amount information as that of the original data $\mathbf{x}$. Furthermore, as we can observe from (1) and (7), $\{z[n]\}$ possesses the same frequency shifting property as that of the original data $\{x[n]\}$ except that now the frequency shifting becomes

$$\begin{aligned} \mathbf{\Gamma} &= (\mathbf{F} e^{-j\omega_c})^{2^I} && (8) \\ &= diag[e^{j(2^I(\omega_1 - \omega_c))}, \cdots, e^{j(2^I(\omega_d - \omega_c))}] && (9) \end{aligned}$$

instead of $\mathbf{F}$ in (1). Therefore, the subspace-based algorithms are also applicable except that now the transformed data $\{z[n]\}$ is being employed rather than $\{x[n]\}$. If we use this new preprocessing scheme in conjunction with the TAM [2], the procedure is now modified as: consider a Hankel matrix $\mathcal{Z}$ which is formed by stacking $\{z[n]\}$ as

$$\mathcal{Z} = \begin{bmatrix} z[0] & z[1] & \cdots & z[N_d - L] \\ z[1] & z[2] & \cdots & z[N_d - L + 1] \\ \vdots & \vdots & \ddots & \vdots \\ z[L-1] & z[L] & \cdots & z[N_d - 1] \end{bmatrix} \tag{10}$$

Using (7), it is easy to show that $\mathcal{Z}$ renders the following decomposition

$$\mathcal{Z} = \mathbf{OQC} \tag{11}$$

where

$$\mathbf{O} = [\mathbf{g}_1^L, \mathbf{g}_2^L, \cdots, \mathbf{g}_d^L]$$

with $\mathbf{g}_i^s = [1, e^{j(\omega_i - \omega_c)2^I}, \cdots, e^{j(\omega_i - \omega_c)2^I(s-1)}]^T$,

$$\mathbf{Q} = diag[q_1, q_2, \cdots, q_d]$$

with $q_i = \frac{1}{\sqrt{2^I}} a_i (1 + e^{j(\omega_i - \omega_c)} + \cdots + e^{j(\omega_i - \omega_c)(2^I - 1)})$, and

$$\mathbf{C} = [\mathbf{g}_1^{N_d - L + 1}, \mathbf{g}_2^{N_d - L + 1}, \cdots, \mathbf{g}_d^{N_d - L + 1}]^T$$

On the other hand, using the singular value decomposition (SVD), $\mathcal{Z}$ can also be factorized as

$$\mathcal{Z} = \mathbf{U}_s \mathbf{\Sigma}_s \mathbf{V}_s^H \tag{12}$$

where $\mathbf{U}_s$, $\mathbf{\Sigma}_s$, and $\mathbf{V}_s$ are $L \times d$, $d \times d$, and $(N_d - L + 1) \times d$ matrices, respectively, and the superscript $^H$ denotes the Hermitian transposition. Using the SRI structure as addressed in [2, 3], it can then be shown that

$$\mathbf{\Gamma} = \mathbf{T} \big( (\mathbf{U}_s^{\downarrow})^{\dagger} \mathbf{U}_s^{\uparrow} \big) \mathbf{T}^{-1} \tag{13}$$

where the superscript $^\dagger$ denotes the Moore-Penrose pseudoinverse, $\mathbf{T}$ is some invertible matrix, and $\mathbf{U}_s^{\uparrow}$ and $\mathbf{U}_s^{\downarrow}$ are derived, respectively, by deleting the first and last rows of $\mathbf{U}_s$. This implies that $\mathbf{\Gamma}$ can be determined by taking the eigendecomposition of $(\mathbf{U}_s^{\downarrow})^\dagger \mathbf{U}_s^{\uparrow}$ and the desired frequencies $\{\omega_i\}_{i=1}^d$ can be easily determined by

$$\omega_i = \frac{Im(\ln \gamma_i)}{2^I} + \omega_c \tag{14}$$

where $\gamma_i$ is the $(i,i)^{th}$ element of $\mathbf{\Gamma}$. To enable the procedure discussed above work properly, the free parameters $I$ and $L$ must be chosen to ensure $\mathbf{W}_P$, $\mathcal{Z}$, $\mathbf{U}_s^{\uparrow}$, and $\mathbf{U}_s^{\downarrow}$ to have rank $d$. It can be verified that a sufficient condition for these rank requirements is

$$\begin{cases} L-1 \geq d, \\ \log_2(\frac{N}{L+d-1}) \geq I \end{cases} \tag{15}$$

### 2.3. Analysis of the Overall Manipulations

The overall effect of the demodulation via (4) together with taking the PSC of the data after the HWT using (5) is as follows. The data vector $\mathbf{x}$ can be rewritten as

$$\mathbf{x} = \mathbf{R}\ \mathbf{h} = [\mathbf{r}_1^N, \mathbf{r}_2^N, \cdots, \mathbf{r}_d^N]\ \mathbf{h} \tag{16}$$

where $\mathbf{r}_i^N = [1, e^{j\omega_i}, \cdots, e^{j\omega_i(N-1)}]^T$. Now, $\mathbf{z} = \mathbf{W}_P\mathbf{P}\mathbf{R}\mathbf{h} = \mathbf{M}\mathbf{h}$, where $\mathbf{M} = \mathbf{W}_P\mathbf{P}\mathbf{R}$. Let $\mathbf{w}_p(k)$ be the $k^{th}$ row of $\mathbf{W}_P$, then the $(k,i)^{th}$ element of $\mathbf{M}$ can be expressed as

$$\mathbf{M}(k,i) = \mathbf{w}_p(k)\ \mathbf{P}\ \mathbf{r}_i^N \tag{17}$$

$$= \begin{cases} \frac{1}{\sqrt{2^I}}\, e^{j(\omega_i-\omega_c)2^I(k-1)}\, \frac{1-e^{j2^I(\omega_i-\omega_c)}}{1-e^{j(\omega_i-\omega_c)}} & \omega_i \neq \omega_c \\ \sqrt{2^I} & \omega_i = \omega_c \end{cases} \tag{18}$$

The magnitude response is

$$\mid \mathbf{M}(k,i) \mid = \begin{cases} \frac{1}{\sqrt{2^I}} \frac{\sin(2^I \frac{\omega_i-\omega_c}{2})}{\sin(\frac{\omega_i-\omega_c}{2})} & \omega_i \neq \omega_c \\ \sqrt{2^I} & \omega_i = \omega_c \end{cases} \tag{19}$$

Therefore the overall effect of demodulation followed by taking the PSC of the transformed data after the WHT is equal to passing the data through a bandpass filter with center frequency at $\omega_c$. From the above equation, we can also observe that the bandwith of the bandpass filter $\mid \mathbf{M}(k,i) \mid$ decreases with the increase of $I$ and is independent of $k$.

### 2.4. The Overall Procedure (in conjunction with the TAM)

The overall algorithm which combines the proposed preprocessing scheme along with the TAM algorithm can be summarized as follows. Do the following steps **in parallel** for all $\omega_c$'s:

Step 1. Perform the demodulation process using (4) with $x[n]$ replaced by the observed noisy data $\tilde{x}[n]$.

Step 2. Determine the PSC of the transformed data based on the HWT using (5).

Step 3. Stack the data computed in step 2 in the structure of (10). Perform the SVD of $\mathcal{Z}$ and use the SRI technique to determine the desired frequencies via (13) and (14).

## 3. SIMULATIONS RESULTS

In this section, we provide some simulations to justify the validity of the proposed algorithm.

**Example**

Consider a set of 32 point 1-D data which contains two harmonics as $(f_1, \phi_1) = (0.52, \pi/4)$ and $(f_2, \phi_2) = (0.50, 0), (\omega_i = 2\pi f_i)$. Two algorithms, the original TAM [2] and the proposed one, have been carried out (For robustness, the "backward version" of data has also been included, i.e. the data matrix $[\ \mathcal{Z},\ \ \mathbf{J}\mathcal{Z}^*\ ]$ is being employed, where $\mathbf{J}$ is an exchange matrix with one's along the antidiagonals and zero's elsewhere, and the superscript $^*$ denotes the complex conjugation.). The parameter chosen for the original TAM is $L = 15$. As for the proposed algorithm, we select $L = 8, 5$, and 3 for $I = 1, 2$, and 3, respectively. The comparisons of the average mean squares errors (MSE's) of $f_1$ and $f_2$ v.s. signal-to-noise ratio (SNR) based on 500 Monte Carlo simulations using these two algorithms are shown in Figs. 2 and 3, respectively.

We can observe that when $I = 1$, the proposed algorithm has almost the same performance as that of the original TAM. For $I = 2$ and 3, the proposed one is slightly inferior with degradation less than 2 d.B. We can also note that the proposed algorithm significantly outperforms the original TAM for low SNR's ($\leq$ 5 d.B). This may be explained by the fact that the spacing between two adjacent frequencies is "amplified" due to the downsampling scheme. Additionally, taking the PSC of the data after the HWT is like putting a "focus" on the desired harmonics and suppressing the noise out of subband of interest. These resolution-enhanced and noise-rejected capabilities become more pronounced for low SNR's.

Note that the computational overhead involved in the demodulation process and the HWT is negligible when compared with that of the SVD. Hence, the overall computational complexity lies mainly in the arithmetic operations required for the SVD. Additionally, the manipulations can be performed in parallel for all $\omega$'s. Consequently, the computations required for the proposed approach is roughly $(\frac{1}{2^I})^3$ of that of the original TAM since the data needed to be processed reduced to $\frac{1}{2^I}$ of the original one, thus leading to substantial reduction of the overall computational complexity. As a result, the proposed algorithm provides an appealing alternative for the harmonic retrieval problem in view of the performance it can offer and the computational overhead it calls for.

## 4. CONCLUSION

A new fast algorithm for the harmonic retrieval is described in this paper. This algorithm relies on a novel preprocessing scheme which first demodulates the input data and then keeps only the PSC of the transformed data after the WHT. Due to the decrease of the data size needs to be further processed, the computational complexity required is thus significantly reduced. Meanwhile, the performance remains roughly the same as that of the original one since the inherent structure of the harmonic data is preserved. Simulation results demonstrate the validity of the proposed approach.

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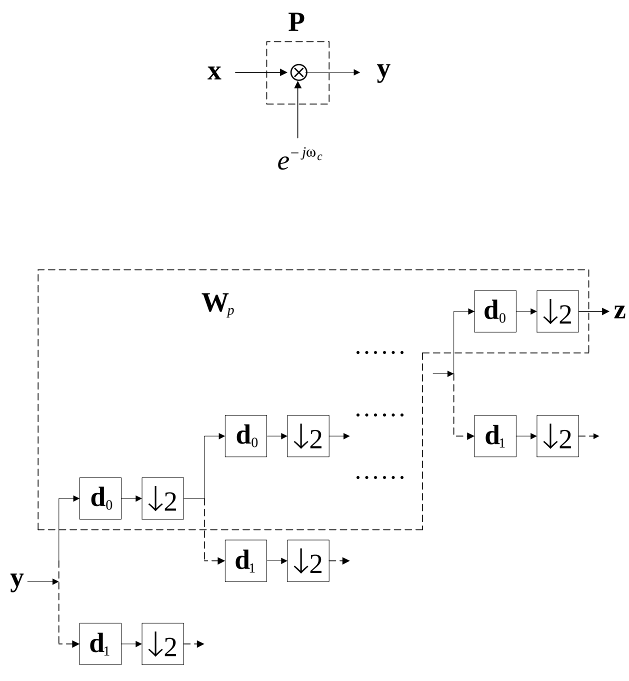


**Figure 1. The block diagram of the proposed preprocessing scheme.**

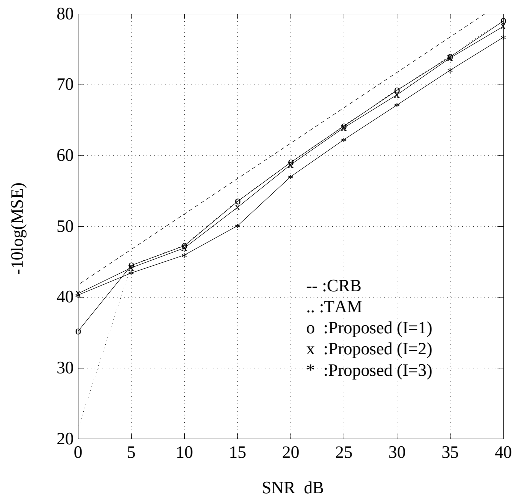


**Figure 2. Comparison of $-10\log(\mathrm{MSE})$ v.s. SNR for the $f_1$ frequency component**

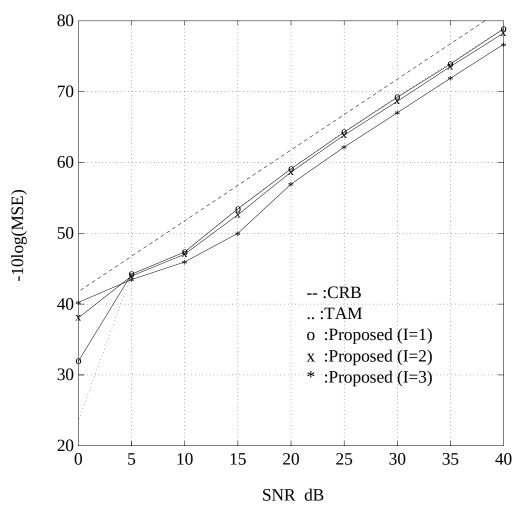


**Figure 3. Comparison of $-10\log(\mathrm{MSE})$ v.s. SNR for the $f_2$ frequency component**