VECTOR SAMPLING EXPANSION: DETERMINISTIC AND STOCHASTIC SIGNALS

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ABSTRACT

This work extends Papoulis' General Sampling Expansion to the vector case where N band limited signals are passed through a multi-input multi-output (MIMO) LTI system that generates M ($M \geq N$) output signals. We find necessary and sufficient conditions for reconstructing the N input signals from the samples of the M output signals, all sampled at N/M the Nyquist rate. A surprising necessary condition is that M/N must be an integer. This condition is no longer necessary when each of the output signals can be sampled at a different rate.

1. INTRODUCTION

In his famous General Sampling Expansion (GSE) [1],[2] Papoulis has shown that a band limited signal $f(t)$ of finite energy, passing through M LTI systems and generates responses $g_k(t), k = 1, \ldots, M$, can be uniquely reconstructed, under some conditions on the M filters, from the samples of the output signals $g_k(nT)$, sampled at $1/M$ the Nyquist rate. More recently [3],[4] the GSE has been extended to multidimensional signals in which the signal f depends on several variables, i.e., $f(\mathbf{x}) = f(x_1, \ldots, x_K)$.

In this work we provide a vector extension of the GSE. We consider N band limited signals (or a signal vector) $\mathbf{f}(t)^T = [f_1(t), \ldots, f_N(t)]$, all having the same bandwidth B, that pass through a multi-input multi-output (MIMO) LTI system, as in Figure 1, to yield M output signals $\mathbf{g}(t)^T = [g_1(t), \ldots, g_M(t)]$ where $M \geq N$. The transfer function of the MIMO system is denoted $\mathbf{H}(\omega)$, where $\mathbf{H}$ is an $M \times N$ matrix, and so we have

$$\mathbf{G}(\omega) = \mathbf{H}(\omega)\mathbf{F}(\omega) \tag{1}$$

where $\mathbf{F}(\omega)^T = [F_1(\omega), \ldots, F_N(\omega)], \mathbf{G}(\omega)^T = [G_1(\omega), \ldots, G_M(\omega)]$ and $F_i(\omega), G_j(\omega)$ are the Fourier transforms of $f_i(t), g_j(t)$ respectively.

The necessary conditions for reconstruction we provide are for signals with no known deterministic functional relationship between them, since dependency between the signals, if known, can be utilized to further reduce the required sampling rate.

We examine whether the N input signals can be reconstructed from samples of the M output signals, at rates that preserves the total rate to be N times the Nyquist rate (the rate obtained by sampling each of the input signals at the Nyquist rate). It turns out, somewhat surprisingly, that there is a distinction between expansion by an integer factor (i.e. M/N is an integer) and expansion by a non-integer factor. When all the output signals are sampled at the same rate (which is N/M times the Nyquist rate) we show that reconstruction of $\mathbf{f}(t)$ is possible, with some conditions on the MIMO system, if and only if M/N is an integer. Reconstruction is also possible when M/N is not an integer, but in this case the sampling rate cannot be equal for all the output signals.

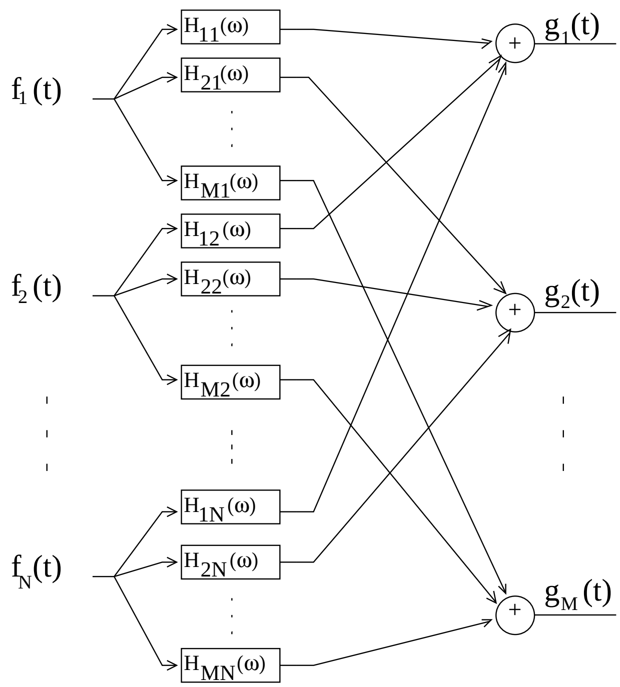

Figure 1. A MIMO LTI system.

2. EXPANSION BY AN INTEGER FACTOR

In this section we consider the case where $M/N = m$ is an integer. We first prove that in this case, under some conditions on the MIMO system, it is possible to reconstruct the inputs from sampling the output at N/M the Nyquist rate.

When sampling at N/M the Nyquist rate, i.e., at a sampling period $T = M\pi/NB$, we get aliased versions of the output signals, which, at the frequency domain, are periodic with a period $c = 2B/m$. We denote by $G_k^a(\omega)$ the Fourier transform of the sampled k-th output signal, and observe that since it is periodic with a period c it is sufficient to consider only one period, say $\omega \in [-B, -B+c]$. At this region, $G_k^a(\omega)$ is composed of m replicas of $G_k(\omega)$, the Fourier transform of the k-th output signal, shifted in frequency by multiples of c, i.e.,

$$G_k^a(\omega) = \frac{c}{2\pi} \sum_{i=0}^{m-1} G_k(\omega + ic), \qquad \omega \in [-B, -B+c] \qquad (2)$$

Since $G_k(\omega) = \sum_{l=1}^{N} H_{kl}(\omega) F_l(\omega)$ where $H_{kl}(\omega)$ is the k, l component of the MIMO system transfer matrix $\mathbf{H}(\omega)$, we have

$$G_k^a(\omega) = \frac{c}{2\pi} \sum_{i=0}^{m-1} \sum_{l=1}^{N} H_{kl}(\omega + ic) F_l(\omega + ic), \omega \in [-B, -B+c] \qquad (3)$$

This is true for $k = 1, 2, \ldots, M$, and so we may write, in a matrix form:

$$\mathbf{G}^a(\omega) = \frac{c}{2\pi} \mathbf{T}(\omega) \mathbf{F}^a(\omega) \quad \omega \in [-B, -B+c] \qquad (4)$$

where $\mathbf{G}^a(\omega)^T = [G_1^a(\omega), G_2^a(\omega), \ldots, G_M^a(\omega)]$, $\mathbf{F}^a(\omega)$ is the M-dimensional vector

$$\begin{aligned} \mathbf{F}^a(\omega)^T \quad = \quad & [F_1(\omega), F_1(\omega + c), \ldots, F_1(\omega + (m-1)c), \\ & \ldots, F_N(\omega), \ldots, F_N(\omega + (m-1)c)] \qquad (5) \end{aligned}$$

i.e., its l-th component $F_l^a(\omega) = F_{l_1}(\omega + (l_2 - 1)c)$ where $l_1 = \lceil l/m \rceil$ and

$$l_2 = \begin{cases} (l \bmod m) & m \text{ does not divide } l \\ m & m \text{ divides } l \end{cases}$$

Finally, $\mathbf{T}(\omega)$ is an $M \times M$ matrix whose k, l component is given by

$$T_{kl}(\omega) = H_{k,l_1}(\omega + (l_2 - 1) \cdot c) \qquad (6)$$

We observe that (4) is a set of M equations for the $mN = M$ unknowns $F_l(\omega + (i-1)c)$, where $l = 1, \ldots, N$ and $i = 1, \ldots, m$. By solving this system of equations we get the Fourier transform of the input signals at all frequencies $\omega \in [-B, B]$, i.e., we can reconstruct the input signals. Note that this system of equations will have a single solution if the determinant of the matrix $\mathbf{T}(\omega)$, which depends solely on the MIMO system, is not zero for every $\omega \in [-B, -B+c]$. Many MIMO systems satisfy this condition, but it should be checked to determine if reconstruction is possible.

One simple example that enables reconstruction is as follows. Let $\mathbf{H}_1$ be an $M \times N$ constant matrix of rank N. If this constant matrix is the MIMO transfer function, reconstruction is impossible, since at any sampling time point we get linear dependent samples. Suppose, however, that we stagger the signals, i.e., shift the k-th output signal by $(k-1)T/M = (k-1)\pi/NB$, and then sample each output signal at sampling period T. This is equivalent to sample at N times the Nyquist rate, while multiplexing between the M output signals. The transfer function of the MIMO system in this case is $\mathbf{H}(\omega) = \mathbf{D}(\omega)\mathbf{H}_1$, where $D(\omega) = diag[1, e^{j\omega T/M}, \ldots, e^{j\omega(M-1)T/M}]$. It is easy to see that in this case the resulting $\mathbf{T}(\omega)$ has a full rank for all ω and so reconstruction is possible.

We next show that we can get such a solvable set of equations for all the frequency content of the input signals only when M/N is an integer, implying that this is a necessary condition for reconstruction.

Suppose M/N is not an integer but then $m < M/N < m+1$ where m is an integer. As we sample, say, the output signal $g_k(t)$ every $T = M\pi/NB$ we get an aliased (sampled) signal whose period in the frequency domain is still $2BN/M$. Again, we choose as the basic period the interval $[-B, -B + 2BN/M]$. This interval can be further divided to N intervals of size $2B/M$ each. We see that in the first ($M \bmod N$) of these N intervals the Fourier transform of the sampled signals, $G_k^a(\omega)$, is composed of $m+1$ replicas of $G_k(\omega)$ while in the rest $N - (M \bmod N)$ intervals there are only m replicas. This situation is illustrated in Figure 2 for the case $N = 2, M = 3$. For the frequencies where there are

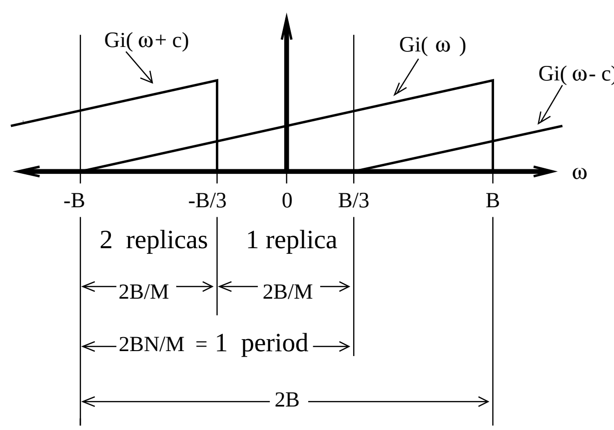

Figure 2. The components of the i-th output channel in the frequency domain when N=2, M=3.

only m replicas we have M equations (an equation for each output signal) for mN unknowns (the unknowns are the m replicas of each of the N input signals). Because $mN < M$ there are more equations than needed for a solution in this interval. This means that we somehow "wasted" samples in this frequency interval, which will cause a "shortage" of samples for the other frequency intervals (because the total sampling rate is exactly N times the Nyquist rate). Indeed, in the frequency intervals where there are $m+1$ replicas, we have $(m+1)N$ unknowns, but only M equations, and since $M < (m+1)N$, the set of $(m+1)N$ equations does not have a single solution, but there is a space of many possible solutions. Since it is assumed that no known functional dependency between the N input signals exists, there are no additional conditions to determine a unique reconstruction of the input signals. Thus, having many possible solutions that are consistent with the measurements implies that we do not have enough information to reconstruct the input signals in this case where M/N is not integer and all outputs are sampled at the same rate.

3. THE INTERPOLATION FORMULA

In this section we provide the explicit formula (the interpolation formula) for reconstruction in the case where $M/N = m$ is an integer and reconstruction is possible. This derivation follows the technique used in [1],[2] and [5].

Let us define $\mathbf{E}_i(t)^T$ to be the M dimensional vector whose k-th component is non-zero and equals $e^{j(k-1-m(i-1))ct}$ only in the region $(i-1)m < k \leq im$ (i.e. at this region it takes the values $1, e^{jct}, \ldots, e^{j(m-1)ct}$), and it is zero elsewhere. Note that since $cT = 2\pi$, $\mathbf{E}_i(t) = \mathbf{E}_i(t-nT)$ for any integer n, i.e., it is periodic with period T.

We now define a set of M-dimensional vectors $\mathbf{Y}_i(\omega, t)^T = [Y_{i,1}(\omega, t), \ldots, Y_{i,M}(\omega, t)]$ as the solution of

$$\mathbf{T}(\omega)^T \mathbf{Y}_i(\omega, t) = \mathbf{E}_i(t) \qquad \omega \in [-B, -B+c] \tag{7}$$

where $\mathbf{T}(\omega)$, defined in (6), is assumed to be invertible at each $\omega \in [-B, -B+c]$ to assure reconstruction. Note that since $\mathbf{E}_i(t)$ is periodic, $\mathbf{Y}_i(\omega, t)$ is also periodic in t with period T.

We also define the signals

$$y_{i,k}(t) = \frac{1}{c} \int_{-B}^{-B+c} Y_{i,k}(\omega, t) e^{j\omega t} d\omega. \tag{8}$$

Note that $y_{i,k}(t)$ is not periodic in t despite the fact that $Y_{i,k}(\omega, t)$ is periodic in t.

We shall prove below that the interpolation formula is given by:

$$f_i(t) = \sum_{k=1}^{M} \left[\sum_{n=-\infty}^{\infty} g_k(nT) y_{i,k}(t - nT) \right] \tag{9}$$

This equation describes a sum of M convolutions of the M sampled sequences with the signals $y_{i,k}(t)$, which are calculated by (8) from the vectors $\mathbf{Y}_i(\omega, t)$ that depend solely on the MIMO system via the relation (7). We can write this result in a matrix form

$$\mathbf{f}(t) = \mathbf{y}(t) * \mathbf{g}^a(t) \tag{10}$$

where $\mathbf{g}^a(t)^T = [g_1^a(t), \ldots, g_M^a(t)]$, $g_k^a(t) = \sum_{n=-\infty}^{\infty} g_k(nT)\delta(t - nT)$, $\mathbf{y}(t)$ is the matrix of signals whose i, k component is $y_{i,k}(t)$, and $*$ here means that convolutions are performed instead of multiplications in the matrix-vector multiplication.

We begin the proof of the interpolation formula by looking at equation (7). This matrix equation can be written as N groups of m equations:

$$\sum_{k=1}^{M} H_{kq}(\omega + uc) Y_{i,k}(\omega, t) = \delta(i - q) e^{juct} \tag{11}$$

where $q = 1, \ldots, N$ and $u = 0, \ldots, (m-1)$. We first discuss the case where $q = i$. Let us expand $Y_{i,k}(\omega, \tau)e^{j\omega\tau}$, considered periodic with period c, in the interval $[-B, -B+c]$ into Fourier series. Using (8) and the periodicity of $Y_{i,k}(\omega, \tau)$ in τ, we see that the coefficients of the expansion are given by the $y_{i,k}(\tau - nT)$'s. We can therefore write

$$Y_{i,k}(\omega, \tau) e^{j\omega\tau} = \sum_{n=-\infty}^{\infty} y_{i,k}(\tau - nT) e^{j\omega nT} \tag{12}$$

Multiplying the m equations generated from (11) choosing $q = i$ by $e^{j\omega t}$ and using (12) we have new m equations

$$\sum_{k=1}^{M} H_{ki}(\omega + uc) \sum_{n=-\infty}^{\infty} y_{i,k}(\tau - nT) e^{j\omega nT} = e^{j(\omega+uc)\tau} \tag{13}$$

This is true for $u = 0, \ldots, (m-1)$ for every $\omega \in [-B, -B+c]$. Using the identity $e^{j\omega nT} = e^{j(\omega+uc)nT}$ and substituting $(\omega + uc)$ for ω, for every u, we conclude that these m equations may be represented by a single equation which is true in the entire interval $[-B, B]$. Thus we now have

$$\sum_{k=1}^{M} H_{ki}(\omega) \sum_{n=-\infty}^{\infty} y_{i,k}(\tau - nT) e^{j\omega nT} = e^{j\omega\tau} \tag{14}$$

for every $\omega \in [-B, B]$. The right side of this equation is the frequency response of an LTI system corresponding to a time shift τ. The left side of the equation describes the sum of many $H_{ki}(\omega) y_{i,k}(\tau - nT) e^{j\omega nT}$ systems. The output of each of the systems on the left side with input $f_i(t)$ is $g_{ki}(t + nT) y_{i,k}(\tau - nT)$, where

$$g_{ki}(t) = h_{ki}(t) * f_i(t) \tag{15}$$

In the time domain we get

$$f_i(t + \tau) = \sum_{k=1}^{M} \left[\sum_{n=-\infty}^{\infty} g_{ki}(t + nT) y_{i,k}(\tau - nT) \right] \tag{16}$$

For the cases where $q \neq i$ we find in a similar way that

$$0 = \sum_{k=1}^{M} \left[\sum_{n=-\infty}^{\infty} g_{kq}(t + nT) y_{i,k}(\tau - nT) \right] \quad , \quad q \neq i \tag{17}$$

From the definition of the MIMO system (1) we recall that

$$g_k(t) = \sum_{q=1}^{N} h_{kq}(t) * f_q(t) = \sum_{q=1}^{N} g_{kq}(t) \tag{18}$$

We now sum equation (16) with the $N-1$ equations of (17) and eventually get

$$f_i(t + \tau) = \sum_{k=1}^{M} \left[\sum_{n=-\infty}^{\infty} g_k(t + nT) y_{i,k}(\tau - nT) \right] \tag{19}$$

Choosing t to be zero and exchanging τ and t leads to the interpolation formula (9).

4. THE STOCHASTIC SIGNAL CASE

We now discuss the interpolation formula for the case where the inputs are N band limited wide-sense stationary (WSS) processes $x_i(t), i = 1, \ldots, N$. We can reconstruct the input process $x_i(t)$ by using the same interpolation equation (9). The reconstructed input $x_i^r(t)$ will be equal in mean square sense to $x_i(t)$, i.e., $E\|x_i(t) - x_i^r(t)\|^2 = 0$. A similar technique to the one used in [5] will be used here. We will follow exactly the steps of the previous section. Equation(14) still describes two LTI systems. Knowing that if two linear systems that have the same frequency response are fed by the same bandlimited WSS input, then the two outputs are equal in the mean square sense ([5] eq.(11-126)), we conclude that if we input a WSS processes $x_i(t)$ instead of the deterministic $f_i(t)$ into the two systems described by equation(14), the two outputs will be equal in the mean sense. Therefore

$$x_i(t+\tau) \stackrel{ms}{=} \sum_{k=1}^{M} \left[\sum_{n=-\infty}^{\infty} g_{ki}(t+nT) y_{i,k}(\tau - nT) \right] \qquad (20)$$

where $g_{ki}(t)$,(which is now a WSS stochastic process), is the output of the $h_{ki}(t)$ LTI filter fed by $x_i(t)$ and the equality is in the mean square sense. Similarly, we can easily reach

$$x_i(t+\tau) \stackrel{ms}{=} \sum_{k=1}^{M} \left[\sum_{n=-\infty}^{\infty} g_k(t+nT) y_{i,k}(\tau - nT) \right] \qquad (21)$$

where the right side equals $x_i^r(t+\tau)$. This concludes the proof for the stochastic signal case.

5. NON-INTEGER EXPANSION FACTOR

For the case where M/N is not an integer, but $L < M/N < L+1$ and L is an integer, we can show that when we sample the i-th output of the MIMO system every $T_i = m_i T$, where m_i is an integer and where $T = \pi/B$, then reconstruction is possible under the condition

$$N = \sum_{i=1}^{M} \frac{1}{m_i} \qquad m_i \text{ integers.} \qquad (22)$$

The proof of this claim, similarly to section 2, is by showing that a set of solvable equations can be generated for the unknown folded spectra of the input signals. The interpolation formula for this case has also been derived and will appear in [6]. Therefore, for every M and N $(M \geq N)$, reconstruction is possible if we sample $N-1$ outputs every $T = \pi/B$, i.e., at Nyquist rate, and sample the rest $M-N+1$ outputs with a time period of $(M-N+1)T$, i.e., at $1/(M-N+1)$ of the Nyquist rate.

In addition, we can find non-LTI MIMO system that expand the number of signal by non-integer factor, and allow reconstruction from the samples of the output signals, all sampled at N/M times the Nyquist rate. One such example is to first modulate the N input signals to non-overlapping frequency bands, and generate one signal whose bandwidth is NB. This signal can then pass through M LTI filters, and by Papoulis' GSE can be reconstructed from samples at $1/M$ its Nyquist rate, which is N/M times the Nyquist rate of the original signals. Other examples of periodically time-varying, linear MIMO system can also be suggested [7].

6. CONCLUSION

Under certain conditions on the sampling rates, it is possible to reconstruct N band limited signals, which are the inputs to of a MIMO LTI system, from periodic samples of the M outputs of the system $(M \geq N)$, with the total average rate of N times the Nyquist rate. Interestingly, not every combination allows reconstruction. Still, at least one allowed sampling combination, in which the sampling periods are multiples of the Nyquist period, always exists.

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