EXTENDING THE CHARACTERISTIC FUNCTION METHOD FOR JOINT a-b AND TIME-FREQUENCY ANALYSIS*

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Abstract—We extend the *characteristic function method* (CFM) to more general groups, operators, and signal spaces. We show that the extended CFM can be applied to projected unitary operators as well as discrete-time/periodic signals.

1 INTRODUCTION AND OUTLINE

The *characteristic function method* (CFM) [1, 2] allows the construction of joint a-b energy distributions[1] $P_x(a,b)$ satisfying *marginal properties*

$$\int_{-\infty}^{\infty} P_x(a,b)\,da = |\langle x, v_b\rangle|^2, \quad \int_{-\infty}^{\infty} P_x(a,b)\,db = |\langle x, u_a\rangle|^2, \tag{1}$$

where $v_b(t)$ and $u_a(t)$ are (generalized [6]) eigenfunctions of self-adjoint operators $\mathcal{B}$ and $\mathcal{A}$, respectively. $\mathcal{B}$ and $\mathcal{A}$ can be associated to $\mathbf{A}_\alpha = e^{j2\pi\alpha\mathcal{B}}$ and $\mathbf{B}_\beta = e^{j2\pi\beta\mathcal{A}}$ ($\alpha, \beta \in \mathbb{R}$) which are unitary representations [7, 8] of the group $(\mathbb{R}, +)$. The eigenfunctions of $\mathcal{B}$, $\mathcal{A}$ equal those of $\mathbf{A}_\alpha$, $\mathbf{B}_\beta$, respectively. A recent extension of the CFM allows α, β to belong to more general LCA groups [9] $(A, \bullet)$, $(B, *)$, respectively, while making the following assumptions [10]–[13]:

1. $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ are unitary representations of the groups $(A, \bullet)$ and $(B, *)$, respectively, i.e., they are unitary operators satisfying $\mathbf{A}_{\alpha_2}\mathbf{A}_{\alpha_1} = \mathbf{A}_{\alpha_1 \bullet \alpha_2}$ and $\mathbf{B}_{\beta_2}\mathbf{B}_{\beta_1} = \mathbf{B}_{\beta_1 * \beta_2}$.

2. $(A, \bullet)$ and $(B, *)$ are isomorphic to $(\mathbb{R}, +)$, and thus also to each other.[2] This implies (i) a correspondence between any "extended" a-b distribution and a "conventional" a-b distribution [12, 13] (i.e., the "extended" CFM is essentially equivalent to the conventional CFM), and (ii) $\mathbf{A}_\alpha = e^{j2\pi\phi(\alpha)\,\mathcal{B}}$ and $\mathbf{B}_\beta = e^{j2\pi\psi(\beta)\,\mathcal{A}}$ where $\mathcal{B}$, $\mathcal{A}$ are self-adjoint with eigenfunctions equal to those of $\mathbf{A}_\alpha$, $\mathbf{B}_\beta$, respectively.

3. The signal space is[3] $L^2(A, d\mu_A)$ or $L^2(B, d\mu_B)$. Hence, all signals $x(t)$ are defined for $t \in (A, \bullet)$ or $t \in (B, *)$.

4. The functions $v_b(t)$, $u_a(t)$ defining the marginals (cf. (1)) are eigenfunctions of $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$, respectively.

This paper looks at the CMF from a new perspective and shows that *all of the above assumptions 1-4 are unnecessary.* This entails a real, essential extension of the CFM that admits much broader classes of operators, groups, and signal spaces. In particular, we shall show that our extended CFM can be applied to "projected" unitary operators and to discrete-time and periodic signals.

2 EXTENDED CFM FOR GROUP $(\mathbb{R}, +)$

For the sake of simplicity, we shall first consider operators $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ indexed by $\alpha, \beta \in (\mathbb{R}, +)$, i.e., assumption 2 is satisfied a priori. (This assumption will be removed in Section 3.) The following theorem shows how two operators $\mathbf{A}_\alpha$, $\mathbf{B}_\beta$ and two function sets $v_b(t)$, $u_a(t)$ must be related for the CMF (based on $\mathbf{A}_\alpha$, $\mathbf{B}_\beta$, $v_b(t)$, $u_a(t)$) to work.

Theorem 1. *Let $\mathcal{X}$ be a signal (Hilbert) space with some inner product $\langle\cdot,\cdot\rangle_{\mathcal{X}}$. Let $\mathbf{A}_\alpha : \mathcal{X} \to \mathcal{X}$ and $\mathbf{B}_\beta : \mathcal{X} \to \mathcal{X}$ be two families of linear operators indexed by $\alpha, \beta \in (\mathbb{R}, +)$. Let $\mathbf{M}_{\alpha,\beta} : \mathcal{X} \to \mathcal{X}$ be a linear operator satisfying*[4]

$$\mathbf{M}_{\alpha,0} = \mathbf{A}_\alpha\,, \qquad \mathbf{M}_{0,\beta} = \mathbf{B}_\beta\,. \tag{2}$$

Let $\phi(\alpha,\beta)$ be a complex-valued function satisfying

$$\phi(\alpha, 0) = \phi(0, \beta) = 1\,. \tag{3}$$

Finally, let $u_a(t)$ and $v_b(t)$ be two families of functions indexed by $a, b \in (\mathbb{R}, +)$. Then, the a-b representation

$$P_x(a,b) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(\alpha,\beta)\,\langle \mathbf{M}_{\alpha,\beta}\,x, x\rangle_{\mathcal{X}}\, e^{-j2\pi(a\beta + b\alpha)}\,d\beta\,d\alpha \tag{4}$$

satisfies the marginal property

$$\int_{-\infty}^{\infty} P_x(a,b)\,da = |\langle x, v_b\rangle_{\mathcal{X}}|^2, \quad \forall\, b \in \mathbb{R} \tag{5}$$

if and only if $\mathbf{A}_\alpha$ is related to $v_b(t)$ as[5]

$$\mathbf{A}_\alpha = \int_{-\infty}^{\infty} (v_b \otimes v_b)\, e^{j2\pi\alpha b}\,db\,. \tag{6}$$

Similarly, $P_x(a,b)$ satisfies the marginal property

$$\int_{-\infty}^{\infty} P_x(a,b)\,db = |\langle x, u_a\rangle_{\mathcal{X}}|^2, \quad \forall\, a \in \mathbb{R} \tag{7}$$

if and only if $\mathbf{B}_\beta$ is related to $u_a(t)$ as

$$\mathbf{B}_\beta = \int_{-\infty}^{\infty} (u_a \otimes u_a)\, e^{j2\pi\beta a}\,da\,. \tag{8}$$

Proof. We shall prove (5); the proof of (7) is analogous. With (4), we have (here, all integrals are from $-\infty$ to ∞)

*Funding by FWF grant P10531-ÖPH.

[1] These can be converted into joint time-frequency energy distributions $P_x(t,f)$ using a mapping $(a,b) \to (t,f)$ [3]–[5].

[2] This assumption has independently been removed in [14].

[3] $L^2(A, d\mu_A)$ is the space of square-integrable functions defined on the set A, with inner product $\langle x, y\rangle = \int_A x(t)\,y^*(t)\,d\mu_A(t)$ where μ_A is the invariant measure for $(A, \bullet)$ [9].

[4] Simple examples of $\mathbf{M}_{\alpha,\beta}$ are $\mathbf{B}_\beta\mathbf{A}_\alpha$ and $\mathbf{A}_\alpha\mathbf{B}_\beta$.

[5] Here $v_b \otimes v_b$ is the linear operator defined by $\big((v_b \otimes v_b)\,x\big)(t) = \langle x, v_b\rangle_{\mathcal{X}}\, v_b(t)$.

$$\int_a P_x(a,b)\,da \;=$$
$$= \int_\alpha \int_\beta \phi(\alpha,\beta)\,\langle \mathbf{M}_{\alpha,\beta}\,x,x\rangle_{\mathcal{X}} \left[\int_a e^{-j2\pi a\beta} da\right] e^{-j2\pi b\alpha}\,d\beta\,d\alpha$$
$$= \int_\alpha \phi(\alpha,0)\,\langle \mathbf{M}_{\alpha,0}\,x,x\rangle_{\mathcal{X}}\, e^{-j2\pi b\alpha}\,d\alpha$$
$$= \int_\alpha \langle \mathbf{A}_\alpha\,x,x\rangle_{\mathcal{X}}\, e^{-j2\pi b\alpha}\,d\alpha \;=\; \left\langle \left(\int_\alpha \mathbf{A}_\alpha\, e^{-j2\pi b\alpha} d\alpha\right) x, x\right\rangle_{\mathcal{X}}$$
$$= \langle (v_b\otimes v_b)\,x,x\rangle_{\mathcal{X}} \;=\; |\langle x, v_b\rangle_{\mathcal{X}}|^2\,,$$

so that (5) is proved. Note that we have used (3), (2), and $\int_\alpha \mathbf{A}_\alpha\, e^{-j2\pi b\alpha}\,d\alpha = v_b\otimes v_b$ (by inversion of (6)). From the above, it is evident that (6) is also necessary for (5). □

While none of assumptions 1,3, and 4 have been used in Theorem 1 or its proof, the central condition in (6) or (8) might be suspected to imply all or some of these assumptions. We now dispel this notion by means of a counterexample.

Projected unitary operators. Let $\mathbf{A}'_\alpha : \mathcal{X}' \to \mathcal{X}'$ and $\mathbf{B}'_\beta : \mathcal{X}' \to \mathcal{X}'$ with $\alpha, \beta \in (\mathbb{R},+)$ be unitary representations of $(\mathbb{R},+)$ on a Hilbert space $\mathcal{X}'$. Let $v'_b(t)$, $u'_a(t)$ be the eigenfunctions of $\mathbf{A}'_\alpha$, $\mathbf{B}'_\beta$, respectively. We make the usual assumption [10, 15] that $v'_b(t)$ and $u'_a(t)$ are complete and orthogonal function sets inducing spectral decompositions

$$\mathbf{A}'_\alpha = \int_{-\infty}^{\infty} (v'_b\otimes v'_b)\, e^{j2\pi\alpha b} db, \quad \mathbf{B}'_\beta = \int_{-\infty}^{\infty} (u'_a\otimes u'_a)\, e^{j2\pi\beta a} da. \tag{9}$$

Now, let $\mathcal{X} \subset \mathcal{X}'$ be some proper subspace of $\mathcal{X}'$, with orthogonal projection operator $\mathbf{P}_{\mathcal{X}}$, and define the "projected operators" $\mathbf{A}_\alpha = \mathbf{P}_{\mathcal{X}}\mathbf{A}'_\alpha\mathbf{P}_{\mathcal{X}}$ and $\mathbf{B}_\beta = \mathbf{P}_{\mathcal{X}}\mathbf{B}'_\beta\mathbf{P}_{\mathcal{X}}$ and the projected eigenfunctions $v_b(t) = \big(\mathbf{P}_{\mathcal{X}} v'_b\big)(t)$ and $u_a(t) = \big(\mathbf{P}_{\mathcal{X}} u'_a\big)(t)$. With (9), we obtain

$$\mathbf{A}_\alpha = \int_{-\infty}^{\infty} \Big(\mathbf{P}_{\mathcal{X}}(v'_b\otimes v'_b)\mathbf{P}_{\mathcal{X}}\Big) e^{j2\pi\alpha b} db = \int_{-\infty}^{\infty} (v_b\otimes v_b)\, e^{j2\pi\alpha b} db, \tag{10}$$

where we have used the identity $\mathbf{P}_{\mathcal{X}}(v'_b\otimes v'_b)\mathbf{P}_{\mathcal{X}} = (\mathbf{P}_{\mathcal{X}} v'_b)\otimes(\mathbf{P}_{\mathcal{X}} v'_b) = v_b\otimes v_b$. Similarly we can show that

$$\mathbf{B}_\beta = \int_{-\infty}^{\infty} (u_a\otimes u_a)\, e^{j2\pi\beta a}\, da\,. \tag{11}$$

While the operators $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ are defined on all of $\mathcal{X}'$, they map $\mathcal{X}$ into $\mathcal{X}$; hence we may (and will, henceforth) redefine them as $\mathbf{A}_\alpha : \mathcal{X}\to\mathcal{X}$ and $\mathbf{B}_\beta : \mathcal{X}\to\mathcal{X}$. Since (10) and (11) are still valid, the conditions (6) and (8) of Theorem 1 are satisfied. Thus, any *a*-*b* representation constructed as in (4) will satisfy the marginal properties (5), (7). We emphasize that $v_b(t)$, $u_a(t)$ are *not* the eigenfunctions, and consequently (10) and (11) are *not* the spectral decompositions, of $\mathbf{A}_\alpha$, $\mathbf{B}_\beta$, respectively (the projected eigenfunctions of an operator are not the eigenfunctions of the projected operator!). The function sets $v_b(t)$ and $u_a(t)$ are complete[6] in $\mathcal{X}$, i.e., $\int_{-\infty}^{\infty}(v_b\otimes v_b)\,db = \int_{-\infty}^{\infty}(u_a\otimes u_a)\,da = \mathbf{I}_{\mathcal{X}}$ (the identity operator on $\mathcal{X}$) but they are not orthogonal, i.e., $\langle v_b, v_{b'}\rangle_{\mathcal{X}} \neq \delta(b-b')$ and $\langle u_a, u_{a'}\rangle_{\mathcal{X}} \neq \delta(a-a')$. Furthermore, $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ are not unitary. They satisfy $\mathbf{A}_0 = \mathbf{B}_0 = \mathbf{I}_{\mathcal{X}}$ but not the usual composition properties, i.e., $\mathbf{A}_{\alpha_2}\mathbf{A}_{\alpha_1} \neq \mathbf{A}_{\alpha_1+\alpha_2}$ and $\mathbf{B}_{\beta_2}\mathbf{B}_{\beta_1} \neq \mathbf{B}_{\beta_1+\beta_2}$. They cannot be written as exponentiated self-adjoint operators $\mathcal{B}$ and $\mathcal{A}$. Finally, $\mathcal{X}$ is generally different from $L^2(A, d\mu_A) = L^2(B, d\mu_B) = L^2(\mathbb{R}, dt)$. Hence, assumptions 1,3, and 4 are indeed violated.

[6] This implies $\int_{-\infty}^{\infty} |\langle x, v_b\rangle_{\mathcal{X}}|^2\, db = \int_{-\infty}^{\infty} |\langle x, u_a\rangle_{\mathcal{X}}|^2\, da = \|x\|^2_{\mathcal{X}}$ for all $x(t) \in \mathcal{X}$. Hence, $|\langle x, v_b\rangle_{\mathcal{X}}|^2$ and $|\langle x, u_a\rangle_{\mathcal{X}}|^2$ are valid 1-D energy distributions with respect to b and a, respectively.

3 EXTENDED CFM FOR GENERAL GROUPS

We now consider the general case $\alpha \in (A,\bullet)$ and $\beta \in (B,*)$, where $(A,\bullet)$ and $(B,*)$ are arbitrary LCA groups that are *not* assumed to be isomorphic to each other or to $(\mathbb{R},+)$, i.e., assumption 2 is not made.

Let $\lambda^A_{\alpha,b}$ and $\lambda^B_{\beta,a}$ denote the group characters of $(A,\bullet)$ and $(B,*)$, respectively [9]. Here $b \in (\tilde{A},\tilde{\bullet})$ and $a \in (\tilde{B},\tilde{*})$ where $(\tilde{A},\tilde{\bullet})$ and $(\tilde{B},\tilde{*})$ are the dual groups of $(A,\bullet)$ and $(B,*)$, respectively [9]. Our central result is formulated as follows.

Theorem 2. *Let $\mathcal{X}$ be a signal (Hilbert) space with some inner product $\langle\cdot,\cdot\rangle_{\mathcal{X}}$. Let $\mathbf{A}_\alpha : \mathcal{X}\to\mathcal{X}$ and $\mathbf{B}_\beta : \mathcal{X}\to\mathcal{X}$ be two families of linear operators indexed by $\alpha \in (A,\bullet)$, $\beta \in (B,*)$. Let $\mathbf{M}_{\alpha,\beta} : \mathcal{X}\to\mathcal{X}$ be a linear operator satisfying*

$$\mathbf{M}_{\alpha,\beta_0} = \mathbf{A}_\alpha\,, \qquad \mathbf{M}_{\alpha_0,\beta} = \mathbf{B}_\beta\,, \tag{12}$$

with α_0, β_0 the identity elements of $(A,\bullet)$, $(B,*)$, respectively. Let $\phi(\alpha,\beta)$ be a complex-valued function satisfying

$$\phi(\alpha,\beta_0) = \phi(\alpha_0,\beta) = 1\,. \tag{13}$$

Finally, let $u_a(t)$ and $v_b(t)$ be two families of functions indexed by $a \in (\tilde{B},\tilde{*})$, $b \in (\tilde{A},\tilde{\bullet})$. Then, the a-b representation[7]

$$P_x(a,b) = \int_A \int_B \phi(\alpha,\beta)\,\langle \mathbf{M}_{\alpha,\beta}\,x,x\rangle_{\mathcal{X}}\, \lambda^{B*}_{\beta,a}\lambda^{A*}_{\alpha,b}\, d\mu_B(\beta)\, d\mu_A(\alpha) \tag{14}$$

(with $a \in (\tilde{B},\tilde{*})$, $b \in (\tilde{A},\tilde{\bullet})$) satisfies the marginal property

$$\int_{\tilde{B}} P_x(a,b)\, d\mu_{\tilde{B}}(a) = |\langle x, v_b\rangle_{\mathcal{X}}|^2, \quad \forall\, b \in (\tilde{A},\tilde{\bullet}) \tag{15}$$

if and only if $\mathbf{A}_\alpha$ is related to $v_b(t)$ as

$$\mathbf{A}_\alpha = \int_{\tilde{A}} (v_b\otimes v_b)\, \lambda^A_{\alpha,b}\, d\mu_{\tilde{A}}(b)\,. \tag{16}$$

Similarly, $P_x(a,b)$ satisfies the marginal property

$$\int_{\tilde{A}} P_x(a,b)\, d\mu_{\tilde{A}}(b) = |\langle x, u_a\rangle_{\mathcal{X}}|^2, \quad \forall\, a \in (\tilde{B},\tilde{*}) \tag{17}$$

if and only if $\mathbf{B}_\beta$ is related to $u_a(t)$ as

$$\mathbf{B}_\beta = \int_{\tilde{B}} (u_a\otimes u_a)\, \lambda^B_{\beta,a}\, d\mu_{\tilde{B}}(a)\,. \tag{18}$$

Proof. With (14), the left-hand side of (15) is

$$\int_{\tilde{B}} P_x(a,b)\, d\mu_{\tilde{B}}(a) \;=\; \int_A \int_B \phi(\alpha,\beta)\,\langle \mathbf{M}_{\alpha,\beta}\,x,x\rangle_{\mathcal{X}} \cdot \left[\int_{\tilde{B}} \lambda^{B*}_{\beta,a}\, d\mu_{\tilde{B}}(a)\right] \lambda^{A*}_{\alpha,b}\, d\mu_B(\beta)\, d\mu_A(\alpha)\,.$$

It follows from the theory of the group Fourier transform[8] that $\int_{\tilde{B}} \lambda^{B*}_{\beta,a}\, d\mu_{\tilde{B}}(a) \stackrel{\triangle}{=} \delta_B(\beta)$ satisfies $\int_B s(\beta)\,\delta_B(\beta)\, d\mu_B(\beta) =$

[7] $\mu_A(\alpha)$, $\mu_B(\beta)$, $\mu_{\tilde{A}}(b)$, and $\mu_{\tilde{B}}(a)$ denote the invariant measures for $(A,\bullet)$, $(B,*)$, $(\tilde{A},\tilde{\bullet})$, and $(\tilde{B},\tilde{*})$, respectively [9].

[8] The group Fourier transform for the group $(B,*)$ is $\tilde{s}(a) = \int_B s(\beta)\,\lambda^{B*}_{\beta,a}\, d\mu_B(\beta)$ with inversion $s(\beta) = \int_{\tilde{B}} \tilde{s}(a)\,\lambda^B_{\beta,a}\, d\mu_{\tilde{B}}(a)$, where $s(\beta) \in L^2(B, d\mu_B)$ and $\tilde{s}(a) \in L^2(\tilde{B}, d\mu_{\tilde{B}})$ [9].

$s(\beta_0)$; hence we obtain further

$$\int_{\tilde{B}} P_x(a,b)\, d\mu_{\tilde{B}}(a) \;=\; \int_A \phi(\alpha,\beta_0)\, \langle \mathbf{M}_{\alpha,\beta_0}\, x, x\rangle_{\mathcal{X}}\, \lambda^{A*}_{\alpha,b}\, d\mu_A(\alpha)$$
$$= \int_A \langle \mathbf{A}_\alpha\, x, x\rangle_{\mathcal{X}}\, \lambda^{A*}_{\alpha,b}\, d\mu_A(\alpha) \;=\; \left\langle \left(\left(\int_A \mathbf{A}_\alpha\, \lambda^{A*}_{\alpha,b}\, d\mu_A(\alpha) \right) x, x \right\rangle_{\mathcal{X}}$$
$$= \langle (v_b \otimes v_b) x, x\rangle_{\mathcal{X}} \;=\; |\langle x, v_b\rangle_{\mathcal{X}}|^2\,,$$

so that (15) is proved. Here we have used (13), (12), and $\int_A \mathbf{A}_\alpha\, \lambda^{A*}_{\alpha,b}\, d\mu_A(\alpha) = v_b \otimes v_b$ (by inversion of (16), cf. Footnote 8). From the above, it is evident that (16) is also necessary for (15). The proof of (17) is analogous. □

We emphasize that none of the assumptions 1-4 have been used in Theorem 2 or its proof. The following example shows that these assumptions may in fact be violated.

Projected unitary operators. Let $\mathbf{A}'_\alpha : \mathcal{X}' \to \mathcal{X}'$ and $\mathbf{B}'_\beta : \mathcal{X}' \to \mathcal{X}'$ be unitary representations of $(A, \bullet)$ and $(B, *)$, respectively. Let $v'_b(t)$, $u'_a(t)$ be the eigenfunctions of $\mathbf{A}'_\alpha$, $\mathbf{B}'_\beta$, respectively, assumed to be complete and orthogonal function sets inducing spectral decompositions

$$\mathbf{A}'_\alpha = \int_{\tilde{A}} (v'_b \otimes v'_b)\, \lambda^A_{\alpha,b}\, d\mu_{\tilde{A}}(b), \quad \mathbf{B}'_\beta = \int_{\tilde{B}} (u'_a \otimes u'_a)\, \lambda^B_{\beta,a}\, d\mu_{\tilde{B}}(a), \tag{19}$$

where the eigenvalues $\lambda^A_{\alpha,b}$, $\lambda^B_{\beta,a}$ are the characters of $(A, \bullet)$, $(B, *)$, respectively. Let $\mathcal{X} \subset \mathcal{X}'$ and define $\mathbf{A}_\alpha = \mathbf{P}_{\mathcal{X}} \mathbf{A}'_\alpha \mathbf{P}_{\mathcal{X}}$, $\mathbf{B}_\beta = \mathbf{P}_{\mathcal{X}} \mathbf{B}'_\beta \mathbf{P}_{\mathcal{X}}$ and $v_b(t) = \big(\mathbf{P}_{\mathcal{X}} v'_b\big)(t)$, $u_a(t) = \big(\mathbf{P}_{\mathcal{X}} u'_a\big)(t)$. With (19) we obtain (cf. the derivation in Section 2)

$$\mathbf{A}_\alpha = \int_{\tilde{A}} (v_b \otimes v_b)\, \lambda^A_{\alpha,b}\, d\mu_{\tilde{A}}(b), \quad \mathbf{B}_\beta = \int_{\tilde{B}} (u_a \otimes u_a)\, \lambda^B_{\beta,a}\, d\mu_{\tilde{B}}(a). \tag{20}$$

Since $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ map $\mathcal{X}$ into $\mathcal{X}$, we will redefine them as $\mathbf{A}_\alpha : \mathcal{X} \to \mathcal{X}$ and $\mathbf{B}_\beta : \mathcal{X} \to \mathcal{X}$. With (20) the conditions (16), (18) are satisfied, and thus any a-b representation constructed as in (14) satisfies the marginal properties (15), (17). Again $v_b(t)$, $u_a(t)$ are *not* the eigenfunctions, and the expressions (20) are *not* the spectral decompositions, of $\mathbf{A}_\alpha$, $\mathbf{B}_\beta$, respectively. The function sets $v_b(t)$, $u_a(t)$ are complete in $\mathcal{X}$ (i.e., $|\langle x, v_b\rangle_{\mathcal{X}}|^2$ and $|\langle x, u_a\rangle_{\mathcal{X}}|^2$ are valid 1-D energy distributions) but not orthogonal. $\mathbf{A}_\alpha$, $\mathbf{B}_\beta$ are not unitary and cannot be written as exponentiated self-adjoint operators. There is $\mathbf{A}_{\alpha_0} = \mathbf{B}_{\beta_0} = \mathbf{I}_{\mathcal{X}}$ but $\mathbf{A}_{\alpha_2} \mathbf{A}_{\alpha_1} \neq \mathbf{A}_{\alpha_1 \bullet \alpha_2}$ and $\mathbf{B}_{\beta_2} \mathbf{B}_{\beta_1} \neq \mathbf{B}_{\beta_1 * \beta_2}$. $\mathcal{X}$ is generally different from $L^2(A, d\mu_A)$ and $L^2(B, d\mu_B)$. Hence, assumptions 1-4 are violated.

4 EXAMPLE: SCALING AND TIME SHIFT OPERATORS ON $L^2([0,T], dt)$

In the following example, assumptions 1, 3, and 4 are violated. Consider the scaling and time-shift operators

$$\mathbf{A}'_\alpha\colon \mathcal{X}'_1 \to \mathcal{X}'_1, \quad (\mathbf{A}_\alpha x)(t) = \frac{1}{\sqrt{\alpha}}\, x\left(\frac{t}{\alpha}\right), \quad \alpha \in (\mathbb{R}_+, \cdot)$$
$$\mathbf{B}'_\beta\colon \mathcal{X}'_2 \to \mathcal{X}'_2, \quad (\mathbf{B}'_\beta x)(t) = x(t-\beta)\,, \quad \beta \in (\mathbb{R}, +),$$

where $\mathcal{X}'_1 = L^2(\mathbb{R}_+, dt)$ and $\mathcal{X}'_2 = L^2(\mathbb{R}, dt)$. $\mathbf{A}'_\alpha$ and $\mathbf{B}'_\beta$ are unitary representations of the LCA groups $(A, \bullet) = (\mathbb{R}_+, \cdot)$ (the group of positive α with multiplication as group operation and invariant measure $d\mu_A(\alpha) = \frac{d\alpha}{\alpha}$) and $(B, *) = (\mathbb{R}, +)$, respectively. $\mathbf{A}'_\alpha$ and $\mathbf{B}'_\beta$ allow (spectral) decompositions (19) with eigenvalues (= group characters) $\lambda^A_{\alpha,b} = e^{j2\pi b \ln\alpha}$ and $\lambda^B_{\beta,a} = e^{-j2\pi\beta a}$ and eigenfunctions $v'_b(t) = \frac{1}{\sqrt{t}}\, e^{-j2\pi b \ln(t/t_0)}$ for $t > 0$ ($t_0 > 0$ is an arbitrary reference time) and $u'_a(t) = e^{j2\pi a t}$, where $b \in (\tilde{A}, \tilde{\bullet}) = (\mathbb{R}, +)$ and $a \in (\tilde{B}, \tilde{*}) = (\mathbb{R}, +)$.

Assume now that we wish to operate on $\mathcal{X} = L^2([0,T], dt)$, the space of square-integrable signals $x(t)$ defined for $t \in [0,T]$, with inner product $\langle x, y\rangle_{\mathcal{X}} = \int_0^T x(t)\, y^*(t)\, dt$. Extending signals as $x(t) = 0$ outside the respective time interval, we have $\mathcal{X} \subset \mathcal{X}'_1 \subset \mathcal{X}'_2$. Unfortunately, $\mathbf{A}'_\alpha$ and $\mathbf{B}'_\beta$ cannot be defined[9] on $\mathcal{X}$ since they may map signals $x(t) \in \mathcal{X}$ onto signals outside $\mathcal{X}$. Hence we use $\mathbf{A}_\alpha = \mathbf{P}_{\mathcal{X}} \mathbf{A}'_\alpha \mathbf{P}_{\mathcal{X}}$ and $\mathbf{B}_\beta = \mathbf{P}_{\mathcal{X}} \mathbf{B}'_\beta \mathbf{P}_{\mathcal{X}}$ which can be considered to be defined on $\mathcal{X}$, i.e., $\mathbf{A}_\alpha : \mathcal{X} \to \mathcal{X}$ and $\mathbf{B}_\beta : \mathcal{X} \to \mathcal{X}$. Here, $(\mathbf{P}_{\mathcal{X}} x)(t) = x(t)\, I(t)$ where $I(t)$ is 1 for $t \in [0,T]$ and 0 otherwise. $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ allow the decompositions (*not* spectral decompositions) (20) with $\lambda^A_{\alpha,b} = e^{j2\pi b \ln\alpha}$, $\lambda^B_{\beta,a} = e^{-j2\pi\beta a}$ as before and $v_b(t) = (\mathbf{P}_{\mathcal{X}} v'_b)(t) = \frac{1}{\sqrt{t}}\, e^{-j2\pi b \ln(t/t_0)}\, I(t)$, $u_a(t) = (\mathbf{P}_{\mathcal{X}} u'_a)(t) = e^{j2\pi a t}\, I(t)$. Hence, conditions (16), (18) are satisfied and joint energy distributions can be constructed according to (14), which yields

$$P_x(a,b) = \int_0^\infty \int_{-\infty}^\infty \phi(\alpha,\beta)\, \langle \mathbf{M}_{\alpha,\beta}\, x, x\rangle_{\mathcal{X}}\, e^{j2\pi(a\beta - b\ln\alpha)}\, d\beta\, \frac{d\alpha}{\alpha}$$

for $a, b \in \mathbb{R}$, where $\phi(\alpha,\beta)$ must satisfy $\phi(\alpha,0) = \phi(1,\beta) = 1$. Setting, for example, $\mathbf{M}_{\alpha,\beta} = \mathbf{B}_\beta \mathbf{A}_\alpha$ yields $\langle \mathbf{M}_{\alpha,\beta}\, x, x\rangle_{\mathcal{X}} = \frac{1}{\sqrt{\alpha}} \int_0^{T+\beta} x\big(\frac{t-\beta}{\alpha}\big)\, x^*(t)\, dt$ for $x(t) \in \mathcal{X}$. $P_x(a,b)$ satisfies the marginal properties (15), (17), which read

$$\int_{-\infty}^\infty P_x(a,b)\, da = |M_x(b)|^2\,, \qquad \int_{-\infty}^\infty P_x(a,b)\, db = |X(a)|^2,$$

with the Mellin-type transform $M_x(b) = \langle x, v_b\rangle_{\mathcal{X}} = \int_0^T x(t)\, e^{j2\pi b \ln(t/t_0)}\, \frac{dt}{\sqrt{t}}$ and the Fourier transform $X(a) = \langle x, u_a\rangle_{\mathcal{X}} = \int_0^T x(t)\, e^{-j2\pi a t}\, dt$.

5 EXTENDED CFM FOR DISCRETE-TIME AND/OR PERIODIC SIGNALS

The removal of assumption 2 in Section 3 allows the extended CFM to be applied to discrete-time and/or periodic signals. (This application has independently been considered in [14].) We shall discuss simple specific examples based on suitably defined time and frequency shift operators.

Discrete-time signals. Let $\mathcal{X} = l^2(\mathbb{Z})$, the space of square-summable discrete-time signals $x(n)$ ($n \in \mathbb{Z}$) with inner product $\langle x, y\rangle_{\mathcal{X}} = \sum_{n=-\infty}^\infty x(n)\, y^*(n)$, and consider the time shift and frequency shift operators

$$\mathbf{A}_\alpha\colon \mathcal{X} \to \mathcal{X}, \quad (\mathbf{A}_\alpha x)(n) = x(n-\alpha), \quad \alpha \in (\mathbb{Z}, +)$$
$$\mathbf{B}_\beta\colon \mathcal{X} \to \mathcal{X}, \quad (\mathbf{B}_\beta x)(n) = e^{j2\pi\beta n} x(n), \quad \beta \in (\mathbb{R}^{\bmod 1}, +_1).$$

$\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ are unitary representations of the LCA groups $(A, \bullet) = (\mathbb{Z}, +)$ and $(B, *) = (\mathbb{R}^{\bmod 1}, +_1)$ (i.e., the interval $[0,1)$ with group operation + mod 1), respectively. These groups are *not* isomorphic to $(\mathbb{R}, +)$, i.e., assumption 2 is violated. They are dual, i.e., the dual group of $(\mathbb{Z}, +)$ is $(\mathbb{R}^{\bmod 1}, +_1)$ and vice versa. $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ allow the (spectral) decompositions $\mathbf{A}_\alpha = \int_0^1 (v_b \otimes v_b)\, \lambda^A_{\alpha,b}\, db$ and

[9] A similar problem occurs in [16], where the time shift operator is considered on $L^2(\mathbb{R}_+, dt)$. Our approach using projected operators explains why the results in [16] are nevertheless valid.

$\mathbf{B}_\beta = \sum_{a=-\infty}^{\infty} (u_a \otimes u_a)\, \lambda^B_{\beta,a}$, with eigenvalues (= group characters) $\lambda^A_{\alpha,b} = e^{-j2\pi\alpha b}$, $\lambda^B_{\beta,a} = e^{j2\pi\beta a}$ and eigenfunctions $v_b(n) = e^{j2\pi bn}$, $u_a(n) = \delta[n-a]$ ($\delta[n-a]$ is 1 for $n = a$ and 0 for $n \neq a$), where $b \in (\tilde{A}, \tilde{\bullet}) = (\mathbb{R}^{\mathrm{mod}\,1}, +_1)$ and $a \in (\tilde{B}, \tilde{*}) = (\mathbb{Z}, +)$. Thus the conditions (16), (18) are satisfied and joint energy distributions can be constructed according to (14), which yields

$$P_x(a,b) = \sum_{\alpha=-\infty}^{\infty} \int_0^1 \phi(\alpha,\beta)\, \langle \mathbf{M}_{\alpha,\beta}\, x, x\rangle_{\mathcal{X}}\, e^{-j2\pi(a\beta - b\alpha)}\, d\beta\,,$$

where $\phi(\alpha,\beta)$ has to satisfy $\phi(\alpha,0) = \phi(0,\beta) = 1$. Setting $\mathbf{M}_{\alpha,\beta} = \mathbf{B}_\beta \mathbf{A}_\alpha$, we obtain $\langle \mathbf{M}_{\alpha,\beta}\, x, x\rangle_{\mathcal{X}} = \sum_{n=-\infty}^{\infty} x(n-\alpha)\, x^*(n)\, e^{j2\pi\beta n}$ and, after simple manipulations,

$$P_x(a,b) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} x(k)\, x^*(l)\, \tilde{\phi}(k-a, l-a)\, e^{-j2\pi(k-l)\,b}$$

with $a \in \mathbb{Z}$ and $b \in [0,1)$, where $\tilde{\phi}(k,l) = \int_0^1 \phi(l-k,\beta)\, e^{j2\pi l\beta}\, d\beta$. This is precisely the "type II Cohen's class" of [17]. $P_x(a,b)$ satisfies the marginal properties (15), (17); with $\langle x, v_b\rangle_{\mathcal{X}} = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi bn} \triangleq X(b)$ and $\langle x, u_a\rangle_{\mathcal{X}} = \sum_{n=-\infty}^{\infty} x(n)\, \delta[n-a] = x(a)$, these marginal properties read

$$\sum_{a=-\infty}^{\infty} P_x(a,b) \;=\; |X(b)|^2, \qquad \int_0^1 P_x(a,b)\, db \;=\; |x(a)|^2.$$

Periodic signals. The dual case ($x(t)$ periodic in time and discrete in frequency) can be treated similarly and leads to the "type III Cohen's class" of [17].

Discrete and periodic signals. Finally let $\mathcal{X} = l^2(\mathbb{Z}_N)$, the space of discrete-time signals $x(n)$ ($n \in \mathbb{Z}_N = \{0, 1, \cdots N-1\}$) which are discrete and finite-support/periodic in both time and frequency domain, with inner product $\langle x, y\rangle_{\mathcal{X}} = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n)\, y^*(n)$. We use the discrete, cyclic time and frequency shift operators [18]

$$\mathbf{A}_\alpha\colon\ \mathcal{X} \to \mathcal{X}, \quad (\mathbf{A}_\alpha x)\,(n) = x((n-\alpha)_N)\,, \quad \alpha \in (\mathbb{Z}_N, +_N)$$

$$\mathbf{B}_\beta\colon\ \mathcal{X} \to \mathcal{X}, \quad (\mathbf{B}_\beta x)\,(n) = e^{j2\pi\frac{\beta n}{N}}\, x(n), \quad \beta \in (\mathbb{Z}_N, +_N),$$

where $(n)_N = n \bmod N$. $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ are unitary representations of the LCA group $(A, \bullet) = (B, *) = (\mathbb{Z}_N, +_N)$, i.e., the set $\mathbb{Z}_N = \{0, 1, \cdots N-1\}$ with group operation + mod N; note that $\int_A s(\alpha)\, d\mu_A(\alpha) = \frac{1}{\sqrt{N}} \sum_{\alpha=0}^{N-1} s(\alpha)$ [18]. The dual group of $(\mathbb{Z}_N, +_N)$ is $(\mathbb{Z}_N, +_N)$. $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ allow the (spectral) decompositions $\mathbf{A}_\alpha = \frac{1}{\sqrt{N}} \sum_{b=0}^{N-1} (v_b \otimes v_b)\, \lambda^A_{\alpha,b}$ and $\mathbf{B}_\beta = \frac{1}{\sqrt{N}} \sum_{a=0}^{N-1} (u_a \otimes u_a)\, \lambda^B_{\beta,a}$, with eigenvalues (= group characters) $\lambda^A_{\alpha,b} = e^{-j2\pi\frac{\alpha b}{N}}$, $\lambda^B_{\beta,a} = e^{j2\pi\frac{\beta a}{N}}$ and eigenfunctions $v_b(n) = e^{j2\pi\frac{bn}{N}}$, $u_a(n) = \sqrt{N}\, \delta[(n-a)_N]$, where $b \in (\tilde{A}, \tilde{\bullet}) = (\mathbb{Z}_N, +_N)$ and $a \in (\tilde{B}, \tilde{*}) = (\mathbb{Z}_N, +_N)$. Thus the conditions (16), (18) are satisfied, and joint energy distributions can be constructed according to (14), which yields

$$P_x(a,b) = \frac{1}{N} \sum_{\alpha=0}^{N-1} \sum_{\beta=0}^{N-1} \phi(\alpha,\beta)\, \langle \mathbf{M}_{\alpha,\beta}\, x, x\rangle_{\mathcal{X}}\, e^{-j2\pi\frac{a\beta - b\alpha}{N}},$$

where $\phi(\alpha,\beta)$ must satisfy $\phi(\alpha,0) = \phi(0,\beta) = 1$. Setting $\mathbf{M}_{\alpha,\beta} = \mathbf{B}_\beta \mathbf{A}_\alpha$, we obtain $\langle \mathbf{M}_{\alpha,\beta}\, x, x\rangle_{\mathcal{X}} = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x((n-\alpha)_N)\, x^*(n)\, e^{j2\pi\frac{\beta n}{N}}$ and finally

$$P_x(a,b) = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} x(k)\, x^*(l)\, \tilde{\phi}((k-a)_N, (l-a)_N)\, e^{-j2\pi\frac{(k-l)b}{N}}$$

with $a, b \in \mathbb{Z}_N$, where $\tilde{\phi}(k,l) = \frac{1}{\sqrt{N}} \sum_{\beta=0}^{N-1} \phi((l-k)_N, \beta)\, e^{j2\pi\frac{l\beta}{N}}$. This is the "type IV Cohen's class" of [17]; it contains the "discrete Wigner distribution" defined in [18]. $P_x(a,b)$ satisfies the marginal properties (15) and (17); with $\langle x, v_b\rangle_{\mathcal{X}} = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi\frac{bn}{N}} \triangleq X_N(b)$ and $\langle x, u_a\rangle_{\mathcal{X}} = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n)\, \sqrt{N}\, \delta[(n-a)_N] = x(a)$, (15) and (17) read

$$\frac{1}{\sqrt{N}} \sum_{a=0}^{N-1} P_x(a,b) \;=\; |X_N(b)|^2, \qquad \frac{1}{\sqrt{N}} \sum_{b=0}^{N-1} P_x(a,b) \;=\; |x(a)|^2.$$

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