OPTIMAL DESIGN OF MULTIRATE SYSTEMS WITH CONSTRAINTS

W. M. Campbell[1] *and T. W. Parks*[2]

[1]Motorola SSTG, Scottsdale, AZ, USA
[2]School of Electrical Engineering, Cornell University, Ithaca, NY, USA

ABSTRACT

The design of constrained multirate systems using a relative ℓ^2 error criterion is considered. A general algorithm is proposed to solve the problem. One application of the algorithm is the design of a new class of multirate filters for signal decomposition–projection filters. These multirate systems are projection operators that optimally approximate linear time-invariant filters in the ℓ^2 norm. A second application of constrained multirate filter design is also presented–optimal design of multistage multirate systems. Examples illustrate the new design method and its advantages over design methods intended for linear time-invariant systems.

1. INTRODUCTION

Constraints are often introduced in multirate design. In implementation of multistage downsampling, *structural* constraints in the system reduce overall required computation [1]. These multiple stages perform most processing at lower rates in order to improve efficiency. In filter bank design, orthogonality and biorthogonality constraints are imposed on the filters in order to achieve perfect reconstruction. These constraints are useful for a variety of applications including image compression, audio compression, and alternating projections [2]. In this paper, we consider the design of multirate systems with either structural constraints on the system or equality/inequality constraints on the design parameters of the system.

The design of multirate systems can be considered from a model-matching approach, see Figure 1. An ideal desired multirate system, $\mathcal{D}$, is to be approximated by a multirate system with FIR filters, $\mathcal{M}(\mathbf{h})$, depending on a vector parameter $\mathbf{h}$. The model-matching system generates an error signal w for a given x. We let x vary over the class of bounded-energy inputs; this choice of signal inputs arises as a natural extension of the Chebyshev criterion for linear time-invariant filter design [3]. The maximum relative system error for this class of inputs is

This work was supported by the National Science Foundation under Grant MIP9224424.

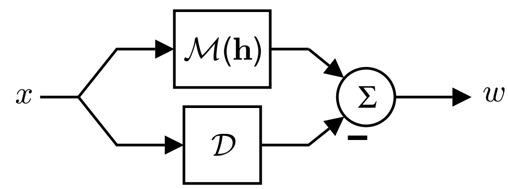

Figure 1: Multirate model-matching problem.

$$\sup_{x\neq 0} \frac{\|w\|_2}{\|x\|_2} = \|\mathcal{M}(\mathbf{h}) - \mathcal{D}\|_2 \qquad (1)$$

where for arbitrary x, $\|x\|_2 = \sqrt{\sum_{n=-\infty}^{\infty} |x(n)|^2}$. For constrained multirate systems, we restrict the model parameters $\mathbf{h}$ to a set $\mathcal{S}$ while minimizing the approximation error (1). The design problem is

$$\hat{\mathbf{h}} = \underset{\mathbf{h}\in\mathcal{S}}{\operatorname{argmin}} \|\mathcal{M}(\mathbf{h}) - \mathcal{D}\|_2. \qquad (2)$$

The model-matching problem for multirate systems can be stated as a matrix-function approximation problem [3, 4]. A general multirate system can be expressed in a commutator form [3] as shown in Figure 2. In the

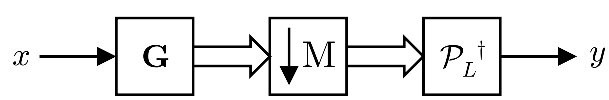

Figure 2: Commutator form of a multirate system.

figure, $\mathbf{G}$ (the commutator-form matrix) is a $L \times 1$ matrix with entries $G_i(z)$, $\mathcal{P}_L^\dagger$ is the adjoint (and inverse) of the L-polyphase decomposition, and the large arrows indicate vector outputs. The polyphase decomposition is defined as

$$[\mathcal{P}_L(x)](n) = \begin{bmatrix} x_0(n) & x_1(n) & \dots & x_{L-1}(n) \end{bmatrix}^t \quad (3)$$

where $x_i(n) = x(Ln+i)$. For an arbitrary $\mathbf{G}$, define $\mathbf{G}_{\text{mod}}$ to be the $L \times M$ matrix with entry (i, m) equal to $G_i(f+\frac{m}{M})$. Then the model-matching problem (2) is (see [3])

$$\hat{\mathbf{h}} = \operatorname*{argmin}_{\mathbf{h}\in\mathcal{S}} \max_{f\in F} \frac{1}{\sqrt{M}} \|\mathbf{E}(f,\mathbf{h})\|_2. \qquad (4)$$

Here $\mathbf{E}(f,\mathbf{h}) = \mathbf{G}_{\rm mod}(f,\mathbf{h}) - \mathbf{G}_{\rm mod}^{\rm ideal}(f)$, $\mathcal{F}$ is the complement of the transition region in $[0,\frac{1}{M}]$, $\mathbf{G}^{\rm ideal}(f)$ is the commutator-form matrix of $\mathcal{D}$, and $\mathbf{G}(f,\mathbf{h})$ is the commutator-form matrix of $\mathcal{M}(\mathbf{h})$. Note that the matrix norm in (4) is the matrix 2-norm [5]; also, note that all matrices are indexed from 0 rather than 1.

2. PROBLEM SOLUTION

The problem (2) can be solved using nonsmooth optimization methods [6]. Define

$$e(\mathbf{h}) = \|\mathcal{M}(\mathbf{h}) - \mathcal{D}\|_2, \qquad (5)$$

then the model-matching problem becomes

$$\hat{\mathbf{h}} = \operatorname*{argmin}_{\mathbf{h}\in\mathcal{S}} e(\mathbf{h}). \qquad (6)$$

Thus, the problem of model matching reduces to minimization of a nonsmooth function on $\mathcal{S}$. The function $e(\mathbf{h})$ is a locally Lipschitz function [7]. For optimization purposes, the generalized gradient, $\partial e(\mathbf{h})$, is needed.

The function $e(\mathbf{h})$ can be expressed as

$$e(\mathbf{h}) = \max_{f\in F} \max_{\mathbf{\Delta}\in B^*} \frac{1}{\sqrt{M}} \mathrm{Re}\Big(\langle \mathbf{E}(f,\mathbf{h}), \mathbf{\Delta}\rangle\Big) \qquad (7)$$

where "Re" indicates the real part, the inner product is given by $\langle A,B\rangle = \sum_{i,k} a_{i,k} b_{i,k}^*$, and B^* is the set of $L\times M$ matrices

$$B^* = \Big\{\mathbf{\Delta} | \sum_{i=0}^{\min(L,M)-1} \sigma_i(\mathbf{\Delta}) = 1\Big\}. \qquad (8)$$

Here, $\sigma_i(\mathbf{\Delta})$ is the ith singular value of the matrix $\mathbf{\Delta}$ [5]. Define

$$\mathcal{S}(\mathbf{h}) = \Big\{(\mathbf{\Delta}, f) | \mathbf{\Delta}\in B^*,\ f\in\mathcal{F}, \frac{1}{\sqrt{M}}\mathrm{Re}(\langle \mathbf{E}(f,\mathbf{h}),\mathbf{\Delta}\rangle) = e(\mathbf{h})\Big\}. \qquad (9)$$

Using the chain rule [7], we obtain

$$\partial e(\mathbf{h}) = \mathrm{co}\Big\{\mathbf{s} | s_i = \frac{1}{\sqrt{M}}\mathrm{Re}\Big(\langle \frac{\partial \mathbf{E}}{\partial h_i}(\mathbf{h},f), \mathbf{\Delta}\rangle\Big), (\mathbf{\Delta}, f)\in\mathcal{S}(\mathbf{h})\Big\}. \qquad (10)$$

For clarity, we mention that $\frac{\partial \mathbf{E}}{\partial h_i}(\mathbf{h},f)$ denotes the evaluation of the partial derivative at $(\mathbf{h},f)$.

In order to add the constraint set to the optimization problem, we use an ℓ^1 penalty function. For instance, for an equality constraint, $f(\mathbf{h}) = 0$, we add the penalty function, $c|f(\mathbf{h})|$ to $e(\mathbf{h})$. Then a minimum of the new objective function $e(\mathbf{h}) + c|f(\mathbf{h})|$ would be a solution. A typical value for the penalty parameter is 10. A subgradient of the new objective function for use in optimization can be found by adding members of the subgradients of objective function and penalty function.

We have implemented in Matlab a nonsmooth optimization method using subgradient locality measures and an implicit trust region strategy as discussed in [6]. At each iteration, the algorithm evaluates the function $e(\mathbf{h})$ and a member of the generalized gradient. More details of the algorithm may be found in [8]. We note that our algorithm is guaranteed only to find a local minimum.

3. EXAMPLES

3.1. Projection Filter Design

We first consider the problem of designing the system shown in Figure 3 with projection constraints on the multirate system. The design problem is to find the best multirate approximation to the ideal filter $H^{\rm ideal}$; we call the resulting multirate system a *projection filter*.

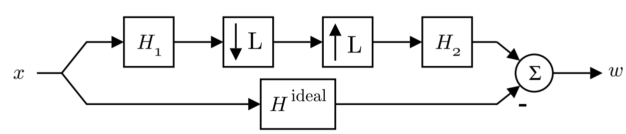

Figure 3: Model-matching for projection filters.

We let $L = 2$ and let the lengths of H_1 and H_2 be $N_1 = N_2 = 101$. We compare the output of a multirate filter to that of a linear phase ideal filter $H^{\rm ideal}$ with a magnitude response in the frequency domain of 1 in $[0, 0.23]$ and 0 in $[0.27, 0.5]$. The design problem is to minimize the relative ℓ^2 output error (1), $\|w\|_2$, subject to the projection (biorthogonality) constraint, $[h_1 * h_2](2n) = 0$, where $'*'$ denotes convolution.

The normed frequency error response,

$$N(f) = \frac{1}{\sqrt{M}} \|E(f,\mathbf{h})\|_2, \qquad (11)$$

of the model-matching system for the optimal design is shown in Figure 4. Shown in Figure 5 are the

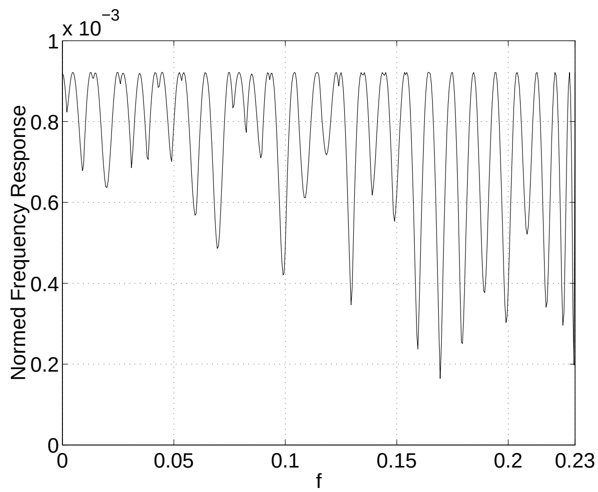

Figure 4: Optimal normed frequency error response for the projection filter.

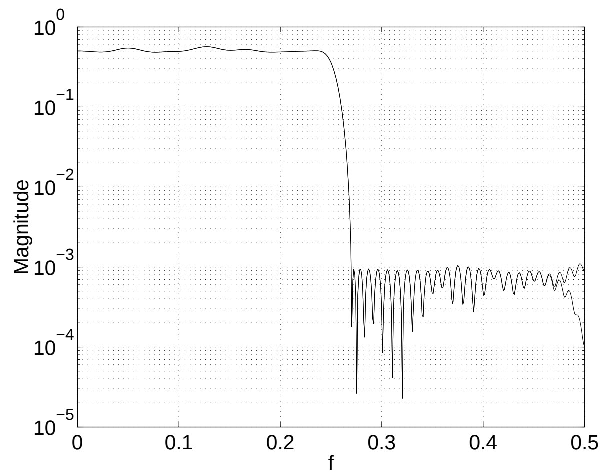

Figure 5: Optimal commutator form filter responses for the projection filter.

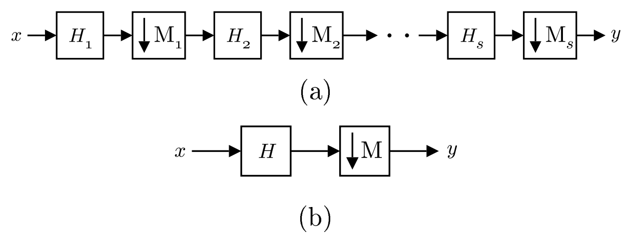

Figure 6: Cascade structure. (a) Multistage form; (b) equivalent single-stage form.

M_1	M_2	N_1	N_2	$e(\mathbf{h}_{\text{optimal}})$	MPS
2	5	3	43	0.0091	29
5	2	10	18	0.0093	19
10	-	85	-	0.0097	42.5

Table 1: Optimal $e(\mathbf{h})$ for various parameters.

responses of the optimal commutator form filters G_1 and G_2 (see Figure 2). Note that the responses overlap significantly. The responses correspond to the filters that are switched between in a commutator fashion.

Projection filters satisfy two main objectives that are important in signal processing. First, the response is a true projection which allows decomposition in terms of the projection subspaces. Second, the system is the best approximation to a linear time-invariant (LTI) filter. This property is unique since it offers a distinct advantage over LTI systems–there is no LTI FIR filter which approximates a lowpass filter that is also a projection.

3.2. Decimator Design

We consider the design of a basic downsampling structure for efficient computation, see Figure 6(a). A simplification of Figure 6(a) is shown in Figure 6(b) where $H(z) = H_1(z)H_2(z^{M_1})\dots H_s(z^{M_1 \dots M_{s-1}})$ and

$$M = M_1 M_2 \dots M_s. \qquad (12)$$

In this case, *structural* constraints have been imposed on the system in Figure 6(b).

We let $M = 10$, require that $e(\mathbf{h}_{\text{optimal}}) \leq 0.01$, and let the ideal filter be linear phase with magnitude 1 in $[0, 0.04]$ and magnitude 0 in $[0.05, 0.5]$. The delay of the ideal filter varies according to the choice of the filter lengths in the approximating system.

Several designs were performed with both one and two stages. Table 1 shows the lowest MPS (multiplications per second) for a given M_1 and M_2 while varying N_1 and N_2 (the lengths of the FIR filters H_1 and H_2). MPS were calculated using the standard methods in [9]; the input sampling rate is 10 Hz (so that the resulting output sampling rate is not fractional). The filters in the approximating system were constrained to be linear phase. The optimal $e(\mathbf{h})$ was found for a fixed M_1 and M_2 using several steps. First, M_1 and M_2 were estimated using the techniques in [9]. Next M_1 was reduced until the optimal $e(\mathbf{h})$ exceeded 0.01. Finally, M_1 and M_2 were adjusted to try to achieve smaller MPS while still having optimal $e(\mathbf{h})$ less than 0.01.

Table 1 shows that $M_1 = 5$ and $M_2 = 2$ is the best choice in terms of MPS. The general rule that M_1 should be chosen greater than M_2 [9] appears to apply to our new design method, although the design methods in [9] were derived under different assumptions.

For comparison, a design with $M_1 = 5$ and $M_2 = 2$ was performed using Chebyshev (Remez) design for each of the filters H_1 and H_2. Filters were designed with don't care bands as in [9]. Repeated designs using the Remez algorithm separately for each H_i were performed to reduce the MPS and still maintain an $e(\mathbf{h})$ less than 0.01. The resulting design has 36.5 MPS, $N_1 = 17$, and $N_2 = 39$. The new method reduces the computation rate by 48 percent over a Chebyshev method. Another comparison was also performed with IFIR filters [10]. For these filters, $N_1 = 17$ and $N_2 = 37$ achieved the design requirement. The computation rate for the IFIR case is 35.5 MPS. In this case, the optimal design results in a 46 percent reduction in computation.

A comparison of the normed frequency error responses, $N(f) = \frac{1}{\sqrt{M}}\|E(f, \mathbf{h})\|_2$, of the model-matching system for the optimal design and the Chebyshev design for $M_1 = 5$, $M_2 = 2$ is shown in Figure 7. Note that $[0.04, 0.05]$ is excluded in the figure since this range is a transition region. The normed frequency error response indicates the error the system makes at a

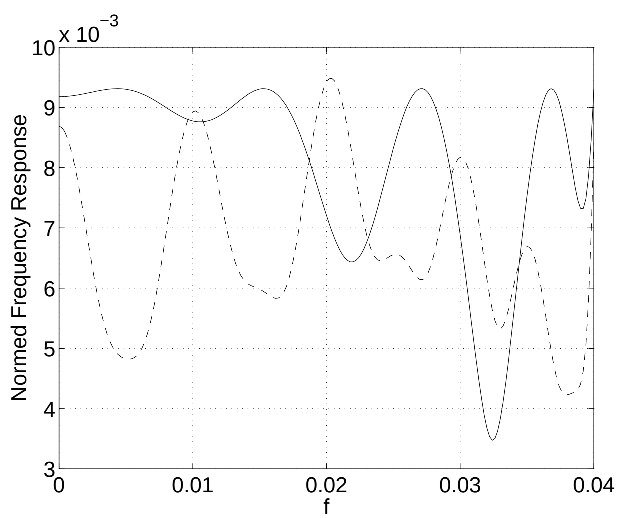

Figure 7: Comparison of optimal design (solid line, 19 MPS) and Chebyshev design (dashed line, 36.5 MPS).

fixed frequency [3]. Note that the optimal design has a normed frequency error response which is equiripple whereas the Chebyshev design has uneven local maxima.

The Chebyshev (Remez) design does not perform well since Chebyshev design optimizes the response of a system for single frequencies; i.e., Chebyshev design optimizes for a worst case input which is concentrated at a single frequency. For multirate design, this single frequency input is not necessarily the worst-case input. Instead, the ℓ^2 model-matching criterion shows that the worst case signals for multirate systems are inputs concentrated at $f, f + 1/M, \ldots, f + (M-1)/M$. Thus, the mismatch of the Chebyshev design method to the worst-case signals creates a suboptimal design.

4. CONCLUSIONS

We have introduced a new method for the design of constrained multirate systems. This method includes the *evaluation* of designs via a relative ℓ^2 error criterion as well as the *optimal design* of systems using nonsmooth optimization methods.

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