DESIGN OF CONJUGATE QUADRATURE FILTERS HAVING SPECIFIED ZEROS

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ABSTRACT

Conjugate quadrature filters with multiple zeros at 1 have classical applications to unitary subband coding of signals using exact reconstruction filter banks. Recent work shows how to construct, given a set of n negative numbers, a CQF whose degree does not exceed 2n-1 and whose zeros contain the specified negative numbers, and applies such filters to interpolatory subdivision and to wavelet construction in Sobelov spaces. This paper describes a recent result of the authors which extends this construction for an arbitrary set of n nonzero complex numbers that contains no negative or negative reciprocal conjugate pairs. Detailed derivations are to be given elsewhere. We design several filters using an exchange algorithm to illustrate a conjecture concerning the minimal degree and we discuss an application to coding transient acoustic signals.

1. INTRODUCTION

For convenience identify filters (finite sequences) with their z-transforms (Laurent polynomials) and let $\mathcal{L}_R$, $\mathcal{L}_N$ denote the set filters whose restriction to the unit circle $T := \{z \in C \,:\, |z| = 1\}$ is real, nonnegative, respectively. In this paper n denotes a positive integer, Λ denotes a set of n (counted with multiplicity) nonzero complex numbers, and $u(\Lambda)$ denotes the subset of the unit disc obtained by replacing the elements in Λ outside the unit disc by their reciprocal conjugates.

Mintzer [1], Smith and Barnwell [2], [3], and Vetterli [4] invented conjugate quadrature filters P, that satisfy

$$|P(z)|^2 + |P(-z)|^2 = 1, \quad z \in T, \tag{1}$$

to perform unitary (lossless) subband coding of discrete signals using exact reconstruction filter banks. Let $Q(\Lambda)$ denote the set of all CQF's whose zeros contain Λ. Recently Micchelli showed if Λ a subset of negative numbers, $Q(\Lambda)$ contains a filter having degree $\leq 2n-1$ and applied these filters to interpolatory subdivision [5] and to wavelet construction in Sobolev spaces [6]. The construction of symmetric CQF's, necessarily having complex coefficients, were described by Lawton in [7], where their utility for handling boundaries in signal coding was demonstrated. In a recent paper [8] we proved:

Result 1 $Q(\Lambda) \neq \phi$ iff

$$(\Lambda \cup \overline{\Lambda}^{-1}) \cap -(\Lambda \cup \overline{\Lambda}^{-1}) = \phi. \tag{2}$$

The only if part follows directly from (1). The if part is proved by constructing a minimal degree $P \in Q(\Lambda)$ as follows: define $S_1(z) := \prod_{\lambda \in \Lambda}(z - \lambda), \;\; z \in C$, and construct filters $P_1, P_2 \in \mathcal{L}_N$ by $P_1(z) := |S_1(z)|^2, \;\; z \in T$, and $P_2(z) := P_1(-z), \;\; z \in C$. Clearly the pair P_1, P_2 satisfies the hypothesis of the following result proved in [8]

Result 2 If the pair $P_1, P_2 \in \mathcal{L}_N$ have no common zeros in $C \backslash \{0\}$, there exists a pair $Q_1, Q_2 \in \mathcal{L}_N$ such that

$$P_1(z)Q_1(z) + P_2(z)Q_2(z) = 1. \tag{3}$$

Choose a pair $Q_1, Q_2 \in \mathcal{L}_N$ whose coefficient sequences have minimal lengths and that satisfy (3). Define $W \in \mathcal{L}_N$ by $W(z) = 1/2(Q_1(z) + Q_2(-z))$. Then construct a spectral factor S_2 of W whose coefficient sequence is supported on the nonegative integers including 0 and define $P := S_1 S_2$. Then (3) implies $P \in Q(\Lambda)$ and has minimal degree. The filter $G := P_1 W$ is *interpolatory* since $G(z) + G(-z) = 1$ and P is a spectral factor of G. Furthermore, if $\Lambda = \overline{\Lambda}$ then W will have real coefficients. Therefore P will have real coefficients if S_2 is chosen to have real coefficients. This will be the case if S_2 is the *minimal phase* (roots are in unit disc) spectral factor of W.

We describe only the basic concept of our proof of Result 2 given in [8]. First, we used standard algebraic methods to construct minimal length filters $B_1, B_2 \in \mathcal{L}_R$ such that

$$P_1 B_1 + P_2 B_2 = 1 \tag{4}$$

and showed that a pair of filters $Q_1, Q_2 \in \mathcal{L}_R$ satisfies (3) if and only if there exists $F \in \mathcal{L}_R$ such that

$$Q_1 = B_1 - F P_2; \quad Q_2 = B_2 + F P_1. \tag{5}$$

Second, we defined rational functions

$$R_1 := -\frac{B_2}{P_1}\,; \quad R_2 := \frac{B_1}{P_2} \tag{6}$$

and showed Q_1 and Q_2 defined by (5) are in $\mathcal{L}_N$ if and only if

$$R_1(z) \leq F(z) \leq R_2(z), \quad z \in T. \tag{7}$$

Third, we used approximation theoretic methods to show the existence of $F \in \mathcal{L}_R$ that satisfies (7).

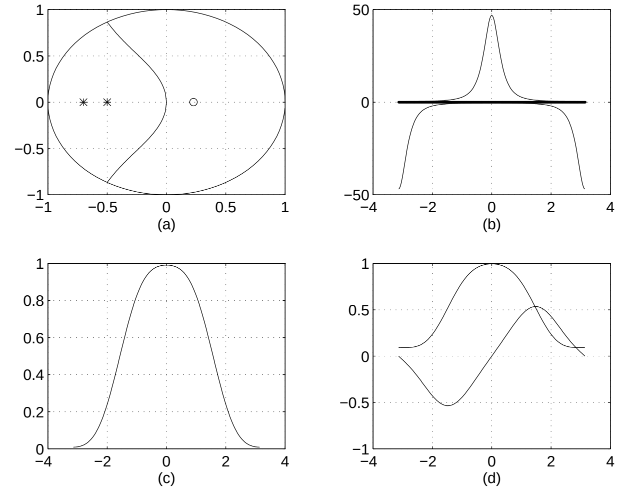

Figure 1. (a) * = 2 Specified Zeros, o = 1 Supplemental Zero. (b) R1, F, R2. (c) Nonnegative Frequency Response of Interpolatory Filter G. (d) Complex Frequency Response of CQF P.

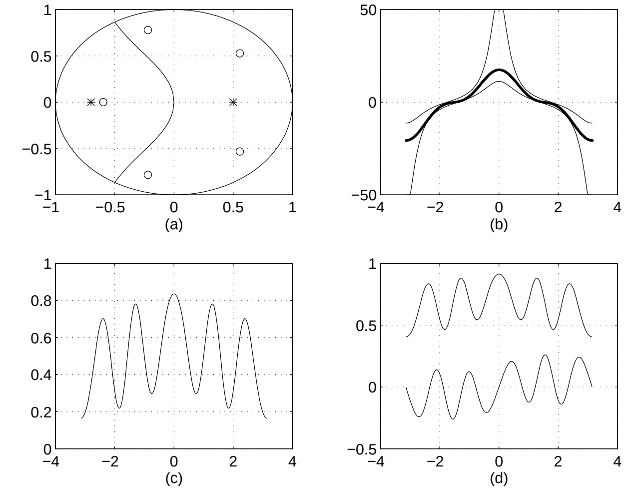

Figure 2. (a) * = 2 Specified Zeros, o = 5 Supplemental Zeros. (b) R1, F, R2. (c) Nonnegative Frequency Response of Interpolatory Filter G. (d) Complex Frequency Response of CQF P.

2. FILTER DESIGN

Using the notation in the previous section define filters $B := B_1P_1 - B_2P_2$ and $D := 2P_1P_2$. Therefore

$$R_1 = \frac{B-1}{D}; \quad R_2 = \frac{B+1}{D} \tag{8}$$

and (7) is equivalent to

$$\max_{z\in T} |B(z) - F(z)D(z)| \leq 1. \tag{9}$$

Clearly if F has minimal degree then Q_1 and Q_2 will have minimal degrees. If D has no roots on the unit circle then the minimal degree F that satisfies (6) may be computed (1) by using the standard Remez exchange algorithm if D and B have real coefficients [9], (2) by using a multiple exchange algorithm if D or B has complex coefficients [10], [11]. If D has roots on the unit circle modifications are required.

Definition The set Λ is called *admissible* if it satisfies (3) and there exists $P \in Q(\Lambda)$ having degree $\leq 2n-1$.

Assume Λ satisfies (2), construct P_1, P_2 as above, and (uniquely) construct minimal length B_1 and B_2 as in [8] so that the filter B has average value 0 over T. The following result is obvious:

Result 3 Λ is admissible if and only if any of the following equivalent conditions hold:

1. $\max_{z\in T} R_1(z) \leq 0 \leq \min_{z\in T} R_2(z)$,
2. the choice $F = 0$ satisfies (7),
3. $|B(z)| \leq 1, \quad z \in T$,
4. the choice $F = 0$ satisfies (9),
5. $u(\Lambda)$ is admissible.

We consider conditions on Λ that ensure admissibility. From the last conditon, we may assume Λ is a subset of the unit disc. The zeros of P that are not in Λ are called *supplemental* zeros of Λ. Define a region $\mathcal{A} \subset C\backslash \{0\}$ by

$$\mathcal{A} := \left\{ x + iy \, : \, r^2 := x^2 + y^2 \leq 1 \, ; \, x < -\frac{r^2}{1+r^2} \right\} \tag{10}$$

Result 4 If $n = 1$, Λ is admissible. If $n = 2$ and $\Lambda = \{\lambda, \overline{\lambda}\}$ is a subset of the unit disc then Λ is admissible iff $\Lambda \subset \mathcal{A}$ or $-\Lambda \subset \mathcal{A}$.

Proof The fact $P(z) := (1 + |\lambda|^2)^{-1}(z - \lambda)$ is a CQF for any nonzero λ proves the first statement. Assume $\Lambda = \{\lambda_1, \lambda_2\}$ with $|\lambda_k| \leq 1$ and $\Lambda = \overline{\Lambda}$. Clearly Λ is admissible iff there exists λ_3 with $|\lambda_3| \leq 1$ and $\alpha > 0$ such that $P(z) := \alpha(z - \lambda_1)(z - \lambda_2)(z - \lambda_3)$ satisfies (1). This is equivalent to $\lambda_1 + \overline{\lambda}_1^{\,-1} + \lambda_2 + \overline{\lambda}_2^{\,-1} + \lambda_3 + \overline{\lambda}_3^{\,-1} = 0$. Clearly, there exists λ_3 satisfying this condition if and only if $\left|\lambda_1 + \overline{\lambda}_1^{\,-1} + \lambda_2 + \overline{\lambda}_2^{\,-1}\right| \geq 2$. If $\lambda_1 = re^{i\theta}$ and $\lambda_2 = re^{-i\theta}$ then this condition becomes $\left|2(r + r^{-1})\cos\theta\right| \geq 2$ which concludes the proof.

Admissibility Conjecture If $\Lambda \subset \mathcal{Z}$ then Λ is admissible.

Figures 1-4 illustrate the design of minimal degree $P \in Q(\Lambda)$ with

$$\begin{aligned} \Lambda \quad = \quad & \{-.7, -.5\}, \; \{-.7, .5\}, \; \{-.2 + .7i, -.2 - .7i\}, \\ & \{-.6 + .8i, -.6 - .8i, -.2 + .3i, -.2 - .3i, -.5\} \end{aligned}$$

respectively. All sets satisfy $\Lambda = \overline{\Lambda}$ and the filters B, D, G, P have real coefficients. The upper left (a) shows the boundary of the unit disc and the set $\mathcal{S}$, the specified roots by $*$, and the supplemental roots by o. The upper

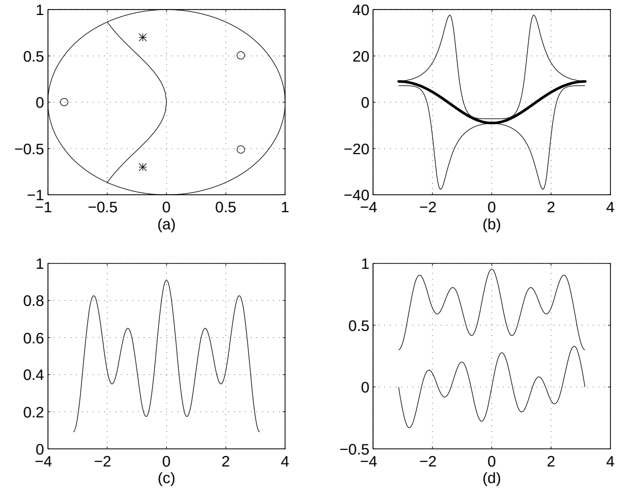

Figure 3. (a) * = 2 Specified Zeros, o = 3 Supplemental Zeros. (b) R1, F, R2. (c) Nonnegative Frequency Response of Interpolatory Filter G. (d) Complex Frequency Response of CQF C.

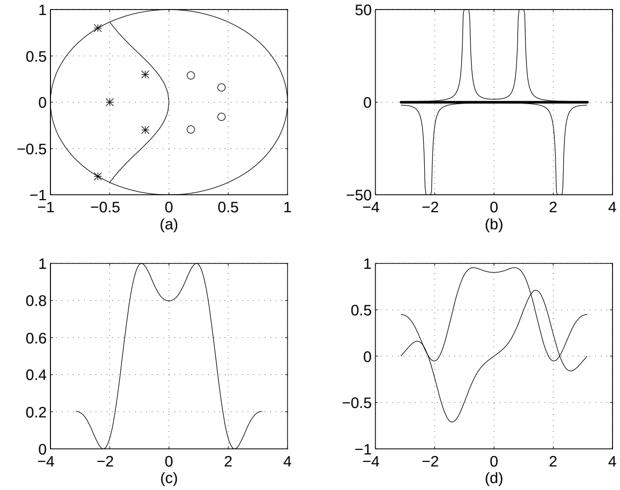

Figure 4. (a) * = 5 Specified Zeros, o = 4 Supplemental Zeros. (b) R1, F, R2. (c) Nonnegative Frequency Response of Interpolatory Filter G. (d) Complex Frequency Response of CQF C.

right (b) plots $R_1 \leq F \leq R_2$ where the smallest length filter F satisfying the inequalities was computed using the MATLAB Remez algorithm. The lower left (c) plots the frequency response of the interpolatory filter $G = P_1 W$ and the lower right (d) plots the real (upper) and imaginary (lower) components of the frequency response of P. In these cases P is the unique minimal phase factor of G and the zeros of P are the union of Λ and the set of supplemental zeros. The fact Λ in (Fig. 1) is admissible while Λ (Fig. 2) and (Fig. 3) are not is implied by Result 4. The fact Λ in (Fig. 4) is admissible supports the Admissibility Conjecture. All the results illustrate the equivalence of the conditions in Result 3.

3. PROCESSING TRANSIENT SIGNALS

Let P be a CQF whose coefficient sequence p is supported on $\{0, 1, \ldots, 2m-1\}$ and define Q, its *twin* CQF, to have coefficient sequence q

$$q_n := (-1)^n \overline{p}_{2m-1-n}, \quad n \in Z. \tag{11}$$

For any sequence s, define sequences $T_p\, s$ and $T_q\, s$ by

$$(T_p s)_k := \sqrt{2} \sum_{n \in Z} \overline{p}_{n-2k} s_n, \quad k \in Z \tag{12}$$

$$(T_q s)_k := \sqrt{2} \sum_{n \in Z} \overline{q}_{n-2k} s_n, \quad k \in Z. \tag{13}$$

Since the sequence q is also a CQF the mapping $s \rightarrow (T_p s, T_q s)$ is unitary (lossless) since for any finitely supported sequence s

$$\sum_{k \in Z} |s_k|^2 = \sum_{k \in Z} |(T_p s)_k|^2 + \sum_{k \in Z} |(T_q s)_k|^2.$$

Therefore, this mapping has an inverse given by its adjoint, and hence s can be *exactly reconstructed* from $T_p s$ and $T_q s$ by the formula

$$s_k = \sqrt{2} \sum_{n \in Z} p_{k-2n} (T_p s)_n + \sqrt{2} \sum_{n \in Z} q_{k-2n} (T_q s)_n, \ \ k \in Z.$$

The design of a CQF with prescribed zeros has potential applications to coding acoustic transient signals. As explained in [8], the mechanical theory of vibrations [12] implies such signals are comprised of linear combinations of functions having the form

$$s_k^j := k^j \, \lambda^k \tag{14}$$

restricted to half intervals of integers $Z \cap [a, \infty]$. Furthermore, as shown in [8], if p is the coefficient sequence of a CQF P having $\overline{\lambda}$ as a zero of multiplicity $j+1$, then

$$(T_h \, s^j)_k = 0 \tag{15}$$

if $2k$ is sufficiently far from the onset a of the transient.

Figure 5 illustrates coding transient signals using CQF's designed to have specified zeros. Part (a) shows a conjugate pair $\Lambda = \{\lambda, \overline{\lambda}\}$ of specified zeros and 1 supplemental zero for a $P \in Q(\Lambda)$ having degree three. Part (b) shows the real signal s defined by

$$s_k := \lambda^k \chi_{[1, infty]}(k) + \overline{\lambda}^k \chi_{[1, infty]}(k), \tag{16}$$

which consists of a sum of functions described by (14). Parts (c) and (d) show the signals $T_p s$ and $T_q s$, where h is the twin CQF sequence defined in (11), and T_p, T_q are defined by (12), (13) respectively. Note the subband coding has essentially *compressed* the signal s into half the number of samples.

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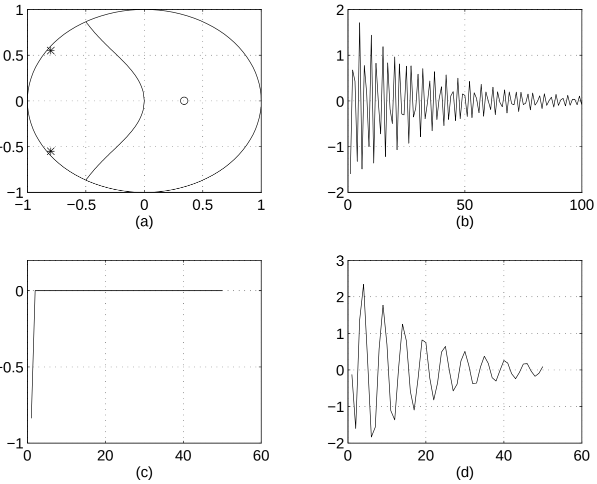

Figure 5. (a) zeros of filter (b) transient acoustic signal corresponding to these zeros, (c) decomposition of suppression band (d) decomposition of passing band.

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