TIME-DOMAIN DESIGN OF LINEAR-PHASE PR FILTER BANKS

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ABSTRACT

In this paper, we present a novel way to design biorthogonal and paraunitary linear phase(LPPUFB) filter banks. The square error of the perfect reconstruction condition is expressed in quadratic form of filter coefficients and the cost function is minimized by solving linear equation iteratively without nonlinear optimization. With some modifications, the method can be extended to the design of paraunitary filter banks. Using this method, we can design LPPUFB with many channels easily and quickly. Design examples are given to validate the proposed method.

1. INTRODUCTION

In recent years, digital filter banks have been well studied from various points of view and are used in various applications in speech, image processing and communications [1]-[3]. In this paper, we consider the maximally-decimated M-channel filter bank. The input signal is separated into M frequency subbands by M analysis filters $H_k(z)$, and the M subband signals are then decimated (by M) to preserve the sampling rate of the system. The resulting M subband signals can be processed (coded, processed, and/or transmitted) and they are combined by interpolators and a set of M synthesis filters $F_k(z)$ to form the reconstructed signal. Filter banks where the reconstructed signal $y(n)$ is a time-delayed version of the input $x(n)$, i.e., $y(n) = cx(n-n_0), c \neq 0$, are called perfect reconstruction filter banks (PRFB). There are many choices of the analysis filter $H_k(z)$ and synthesis filter $F_k(z)$ that will satisfy the perfect reconstruction conditions. However, for image processing applications, it is important for $H_k(z)$ and $F_k(z)$ to be linear phase. If the analysis filters have linear phase, all the subband signals are delayed by the same group delay. Moreover, linear-phase filters allow us to use simple symmetric extension methods to accurately handle finite-length signals' boundaries[4]. In this paper, we only discuss linear phase perfect reconstruction filter banks (LPPRFB). Several design methods of M-channel LPPRFB have been reported [?]-[6]. Many of the reported methods use lattice structure which consists of orthogonal matrices as building blocks. These design methods require nonlinear optimization which is computational complex and sensitive to initial values. Authors have presented a design method of LPPRFB without nonlinear optimization[7]. This method is based on cancellation of all distortions occuring in FB. The cost function is expressed as minimization problem of all distortions in frequency domain and formulated as quadratic form of filter coefficients. Although nonlinear optimization is not needed, the design time is long since we have to compute the integration in frequency domain.

In this paper, a new design of LPPRFB based on time-domain constraints is presented. PR condition using polyphase matrix can be transformed into time-domain conditions. Cost function is defined as PR constraints such that time-domain constraints are minimized and it can be written as quadratic functions of filter coefficients vectors of both analysis and synthesis systems. This method solves a set of linear equations iteratively without the use of nonlinear optimization. The most significant difference between the proposed method and the conventional ones is to minimize least square error of PR constraints in time-domain instead of frequency domain. Since the objective function is in quadratic form, one can iteratively solve for individual variables while keeping the other as constant. The overall error can be shown to be monotonously decreasing and an optimal solution can be obtained. This method is modified to design paraunitary filter bank by observing that the synthesis filters are time-reversed version of the analysis filters.

2. PRELIMINARIES

Fig.1 shows a maximally-decimated M-channel filter bank where $H_k(z)$ and $F_k(z)$ denote analysis and synthesis filters, respectively.

The analysis and synthesis filters are expressed in polyphase component form, using type-1 polyphase filters for the analysis and type-2 polyphase filters for the synthesis,

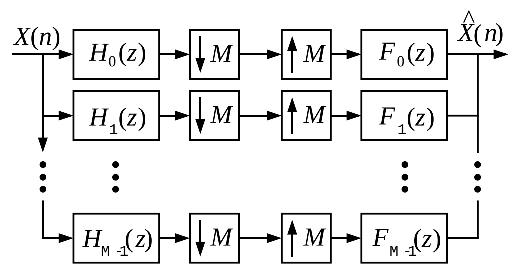

Figure 1. Maximally-decimated M-channel filter bank.

$$
\begin{aligned}
H_k(z) &= \sum_{j=0}^{M-1} z^{-j} E_{k,j}(z^M) \\
F_k(z) &= \sum_{j=0}^{M-1} z^{-(M-1-j)} R_{j,k}(z^M)
\end{aligned}
$$

We define the polyphase matrix of analysis and synthesis bank as $[\mathbf{E}(z)]_{k,j} = E_{k,j}(z)$ and $[\mathbf{R}(z)]_{k,j} = R_{k,j}(z)$. Then the perfect reconstruction condition can be defined using polyphase matrices as

$$
\mathbf{R}(z)\mathbf{E}(z) = z^{-p}\mathbf{I}, \quad p > 0 \tag{1}
$$

Time-domain constraints of PRFB is defined by

$$
\sum_{n=0}^{N-1} h_k(n) f_m(n - M\ell) = \delta(k-m)\delta(\ell) \tag{2}
$$

In paraunitary case, the synthesis filters are chosen as the time reversed versions of the analysis filters

$$
F_k(z) = z^{-(N-1)} H_k(z^{-1}) \quad 0 \le k \le M-1 \tag{3}
$$

Here, $\mathbf{R}(z) = \tilde{\mathbf{E}}(z) = \mathbf{E}^T(z^{-1})$ and the PR condition becomes

$$
\tilde{\mathbf{E}}(z)\mathbf{E}(\mathbf{z}) = \mathbf{I}. \tag{4}
$$

Similar to the nonparaunitary case, time-domain condition of PRPUFB is expressed by

$$
\sum_{n=0}^{N-1} h_k(n) h_m(n - M\ell) = \delta(k-m)\delta(\ell) \tag{5}
$$

In this paper, we show a new design method of PRFB based on the time-domain constraints as mentioned above.

There exists solution to Eq.(7) iff Theorem 1 is satisfied by all the analysis filters [?][8].

Theorem 1: For an M-channel linear-phase perfect reconstruction filter banks with filter lengths $N_i = K_i M + \beta,\ 0 \le \beta < M,\ K_i \ge 1,$ the symmetric conditions are:

1. If M is even and β is even, there are $\frac{M}{2}$ symmetric, and $\frac{M}{2}$ antisymmetric filters.
2. If M is odd, there are $(\frac{M+1}{2})$ symmetric and $(\frac{M-1}{2})$ antisymmetric filters.

The length conditions are:

1. If M is even and β is even, $\sum_{i=0}^{M-1} K_i$ is even.
2. If M is odd and β is even, $\sum_{i=0}^{M-1} K_i$ is odd.
3. If M is odd and β is odd, $\sum_{i=0}^{M-1} K_i$ is even.

3. DESIGN OF LINEAR-PHASE PERFECT RECONSTRUCTION FILTER BANKS

In this section, we present a new design method for biorthogonal LP filter banks, where the PR condition is expressed in quadratic form of filter coefficients

3.1. Biorthogonal Filter Banks

A. Cost Function (even M)

In the proposed method, the least square error of the PR condition expressed by Eq.(2) is minimized. A cost function Φ is defined as

$$
\begin{aligned}
\Phi &= \sum_{j=0}^{M-1} \left[\sum_{n=0}^{N-1} f_j(n) h_j(n) - \frac{1}{M} \right]^2 \\
&+ \sum_{\substack{\ell=0 \\ \neq(\ell=0, i=j)}}^{\lfloor (N-1)/M \rfloor} \sum_{j,k=0}^{M-1} \left[\sum_{n=0}^{N-1} f_j(n) h_k(n - \ell M) \right]^2 \\
&= \Phi_a + \Phi_b
\end{aligned} \tag{6}
$$

The first term denotes the orthogonality condition and the second term corresponds to the shift-orthogonality condition. The above equation can be rewritten in matrix form as

$$
\sum_{n=0}^{N-1} f_j(n) h_k(n - \ell M) = \hat{\mathbf{f}}_j^T \mathbf{Q}_\ell \hat{\mathbf{h}}_k \tag{7}
$$

where $\hat{\mathbf{f}}_j$ and $\hat{\mathbf{h}}_k$ are vectors of coefficients from $F_k(z)$ and $H_k(z)$, respectively and $\mathbf{Q}_\ell$ is a square matrix of size N as

$$
\mathbf{Q}_\ell = \begin{array}{c} M\ell \, \{ \\ \\ \\ \\ \end{array}
\begin{bmatrix}
\mathbf{0} & & \cdots & \mathbf{0} \\
1 & 0 & & \mathbf{0} \\
 & \ddots & & \\
\mathbf{0} & & 1 & \mathbf{0}
\end{bmatrix} \tag{8}
$$

The case of $\ell = 0$ are referred to as the orthogonality conditions; the remaining cases are referred to as the shift-orthogonality conditions. Since the analysis and synthesis filters have linear phase and the filter length of the even channel filter banks must be even according to Theorem 1, their impulse response with even length are expressed by half of the number of coefficients as:

$$
\hat{\mathbf{h}}_i = \begin{bmatrix} \mathbf{I} \\ (-1)^i \mathbf{J} \end{bmatrix} \mathbf{h}_i = \mathbf{W}_i \mathbf{h}_i \tag{9}
$$

where $\mathbf{h}_i$ is the vector with half of the number of coefficients and $\mathbf{J}$ is the anti-diagonal matrix. The filter with even index is symmetric and the odd-indexed one is anti-symmetric. With these matrices, the first term of Eq.(6) can be rewritten as,

$$
\begin{aligned}
\Phi_a &= \sum_{i=0}^{M-1} \left\{ \mathbf{f}_i^T \mathbf{W}_i^T \mathbf{Q}_0 \mathbf{W}_i \mathbf{h}_i - \frac{1}{M} \right\}^2 \\
&= \sum_{i=0}^{M-1} \Big\{ \mathbf{h}_i^T \mathbf{Q}_{0,i,i}^T \mathbf{f}_i \mathbf{f}_i^T \mathbf{Q}_{0,i,i} \mathbf{h}_i \\
&- \frac{2}{M} \mathbf{f}_i^T \mathbf{Q}_{0,i,i} \mathbf{h}_i + \frac{1}{M^2} \Big\}
\end{aligned} \tag{10}
$$

where

$$
\mathbf{Q}_{0,i,j} = \mathbf{W}_i^T \mathbf{Q}_0 \mathbf{W}_j = \begin{cases} 2\mathbf{I} & , \quad i + j = 2p \\ \mathbf{0} & , \quad otherwise \end{cases} \tag{11}
$$

The second term Φ_b can be expressed similarly as

$$\Phi_b = \sum_{\substack{\ell=0 \\ \neq(\ell=0,i=j)}}^{\lfloor (N-1)/M \rfloor} \sum_{i,j=0}^{M-1} \left\{ \mathbf{h}_j^T \mathbf{Q}_{\ell,i,j}^T \mathbf{f}_i \mathbf{f}_i^T \mathbf{Q}_{\ell,i,j} \mathbf{h}_j \right\}. \quad (12)$$

The overall cost function $\Phi = \Phi_a + \Phi_b$ can be written as follows,

$$\begin{aligned} \Phi &= \sum_{\ell=0}^{\lfloor (N-1)/M \rfloor} \sum_{i,j=0}^{M-1} \left\{ \mathbf{h}_j^T \mathbf{Q}_{\ell,i,j}^T \mathbf{f}_i \mathbf{f}_i^T \mathbf{Q}_{\ell,i,j} \mathbf{h}_j \right\} \\ &- \frac{2}{M} \sum_{i=0}^{M-1} \mathbf{f}_i^T \mathbf{Q}_{0,i,i} \mathbf{h}_j + \frac{1}{M} \end{aligned} \quad (13)$$

If $\mathbf{f}_i$ is fixed, the cost function is regarded as quadratic form of filter coefficient vector of the analysis filter. The minimization of this error is achieved when $\frac{\partial \Phi}{\partial \mathbf{h}_i} = \mathbf{0}$:

$$\begin{aligned} \frac{\partial \Phi}{\partial \mathbf{h}_j} &= 2\mathbf{h}_j^T \left\{ \sum_{\ell=0}^{\lfloor (N-1)/M \rfloor} \sum_{i=0}^{M-1} \mathbf{Q}_{\ell,i,j}^T \mathbf{f}_i \mathbf{f}_i^T \mathbf{Q}_{\ell,i,j} \right\} \\ &- \frac{4}{M} \mathbf{f}_j^T \\ &= 2\mathbf{h}_j^T \mathbf{P}_j - \frac{4}{M} \mathbf{f}_j^T = \mathbf{0} \end{aligned} \quad (14)$$

where we have used $\mathbf{Q}_{0,i,i} = 2\mathbf{I}$. $\mathbf{P}_j$ is a square matrix of size $N/2$ and $\mathbf{P}_0 = \mathbf{P}_{2j}$, $\mathbf{P}_1 = \mathbf{P}_{2j+1}$ since $\mathbf{W}_0 = \mathbf{W}_{2j}$ and $\mathbf{W}_1 = \mathbf{W}_{2j+1}$. $\mathbf{h}_j$ can be obtained by calculating only two inverse matrices of sizes $N/2$ independent of the number of channels as

$$\mathbf{h}_j = \frac{2}{M} \mathbf{P}_{(j \ mod \ 2)}^{-T} \mathbf{f}_j \quad (15)$$

Although one can not prove that $\mathbf{P}^{-1}$ exists, it is always true in all of our design examples. However, form experimentation we have found that it exist. Similary, one can obtain $\mathbf{f}_i$ by fixing $\mathbf{h}_j$.

B. Design algorithm of LP biorthogonal FB

The design procedure is as follows;

1. Design analysis filters using any suitable method such as LMS or remez algorithm.
2. Fix the coefficients of analysis filters $\mathbf{h}_i$ and then, calculate synthesis filters $\mathbf{f}_j$ from Eq.(15)
3. Fix the coefficients of synthesis filters $\mathbf{f}_j$ to the value obtained by the above equation. Then, calculate analysis filters $\mathbf{h}_i$ in the same way.
4. Terminate if the cost function Φ is small enough($\Phi \leq 10^{-10}$), otherwise continue the algorithm (back to step 2)

The above algorithm yields biorthogonal filter banks and it only requires solving linear equation iteratively. Convergence of the algorithm is guaranteed and cost function can be shown to be monotonously decreasing. [7]

In our method, the cost function does not impose any constraints in the frequency response of the analysis and synthesis filters. However, if the intial frequency responses of the analysis filters have good frequency selectivity, the resulting synthesis filters must also have good responses for aliasing cancellation. Therefore if we design the initial analysis filter with suitable property, filter banks with good response can be obtained and are independent of the initial values. This has been confirmed by design examples.

C. Cost function (odd M)

Design procedure of odd-channel LPPRFB is similar to even-channel one. Since the filter length of each filters must be odd in odd-channel, the impulse responses are rewritten by

$$\begin{aligned} \hat{\mathbf{h}}_{2i} &= \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & 1 \\ \mathbf{J} & \mathbf{0} \end{bmatrix} \mathbf{h}_{2i} = \mathbf{W}_{2i}^o \mathbf{h}_{2i} \\ \hat{\mathbf{h}}_{2i+1} &= \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \\ -\mathbf{J} \end{bmatrix} \mathbf{h}_{2i+1} = \mathbf{W}_{2i+1}^o \mathbf{h}_{2i+1} \end{aligned}$$

with the following relation

$$\begin{aligned} &\mathbf{Q}_{0,i,j}^o = \mathbf{W}_i^{oT} \mathbf{Q}_0 \mathbf{W}_j^o \\ &= \begin{cases} \begin{bmatrix} 2\mathbf{I} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} & , \quad i+j = 2p, i : even \\ 2\mathbf{I} & , \quad i+j = 2p, i : odd \\ \mathbf{0} & , \quad otherwise \end{cases} \end{aligned} \quad (16)$$

where $\mathbf{h}_{2i}$ and $\mathbf{h}_{2i+1}$ are vectors with size $(N+1)/2$ and $(N-1)/2$, respectively. Substituting these equations into Eq.(14), same procedure as in the even case is applicable to the design of LPPRFB with odd M.

3.2. LP Paraunitary Filter Banks

The main difference between paraunitary and biorthogonal FB is in the choice of synthesis system. In paraunitary case, synthesis system is time reversed version of the analysis bank which implies that

$$\begin{aligned} h_k(n) &= f_k(n) \qquad \text{for symmetry} \\ h_k(n) &= -f_k(n) \qquad \text{for antisymmetry} \end{aligned}$$

One can use the same design algorithm as in the previous section and the only difference is that the coefficient vector is chosen to be the average of $h_k(n)$ and $f_k(n)$ in each iteration. In other words, the average value of analysis and synthesis filters is used for $\mathbf{f}_k$ instead of solving for $\mathbf{f}_k$ in the step 3 of the design algorithm.

$$\mathbf{f}_k^{(i+1)} = (\mathbf{h}_k^{(i)} + \mathbf{f}_k^{(i)})/2 \quad (17)$$

B. M-channel orthonormal and symmetric wavelet with K-regularities

It has been shown that M band orthonormal wavelets are characterized by the unitary scaling function $H_0(z)$. The unitary scaling function is K-regular if the lowpass filter $H_0(z)$ has the form

$$H_0(z) = \left(\frac{1 + z^{-1} + \cdots + z^{-(M-1)}}{M} \right)^K \hat{H}(z) \quad (18)$$

The impulse response of $H_0(z)$ is expressed by

$$\mathbf{h}_0 = (\frac{1}{M}\mathbf{u}*)^K \hat{\mathbf{h}}_0 \tag{19}$$

where $(*)$ denotes convolution and $\mathbf{u}$ is the step response of size M. The above equation is rewritten using the matrix form as

$$\mathbf{h}_0 = \left(\prod_{k=1}^{K} \mathbf{U}_k\right) \hat{\mathbf{h}}_0 \tag{20}$$

where $\mathbf{U}_k$ is $(\hat{N}+(M-1)k)\times(\hat{N}+(M-1)(k-1))$ matrix that corresponds to the convolution matrix with step response of size M. The remaining filters except for lowpass filter are same as that in Eq.(9). In this case, we need to find three inverse matrices.

4. DESIGN EXAMPLES

In this section, we show some design examples of non-paraunitary and paraunitary filter banks. GUI (Graphical user interface) software to design LPPRFB can be found at url address *http://saigon.ece.wisc.edu/~waveweb/QMF.html* under SOFTWARE and Linear-Phase near PR Filter Bank Design.

A.30 channel LP paraunitary filter bank with PMI
30-channel LPPUFB with length 60 is designed. We have imposed the PMI property on this filter bank. Fig.2 shows the frequency responses of the analysis and synthesis filters. This results are obtained after 7 iterations in 3.86 sec and the PR error is 10^{-10}.

B. 9-channel LP paraunitary filter bank with 1-regularity
9-channel LPPUFB with 1-regularity is designed. The length of each analysis filter is 17. Fig.3 shows the frequency responses of the analysis and synthesis filters. This results are obtained after 24 iterations in 3.19 sec and the PR error is 10^{-10}.

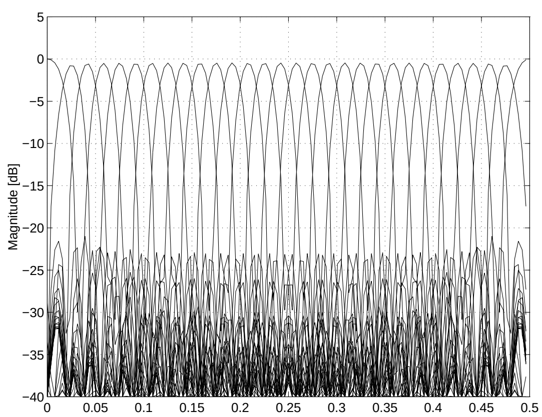

Figure 2. 30-channel paraunitary filter bank with length 60.

5. CONCLUSION

In this paper, we propose a time-domain approach to design biorthogonal and paraunitary filter banks with linear phase. Instead of applying nonlinear optimization, the proposed method solves a set of linear equations iteratively where PR error is minimized instead of imposing it as a design constraint. As a result, a more efficient method is obtained when compared to LPPRFB design method in frequency domain because we do not need to calculate the integral. LPPUFB with arbitrary number of channel and length satisfying Theorem 1 can be designed using the same algorithm but averaging the analysis and synthesis filter coefficients. The proposed method can be extended to the design of LPPRFB with pairwise mirror image property and wavelets with K-reguralities by imposing such properties as design constraints on the transfer functions of analysis and synthesis filters.

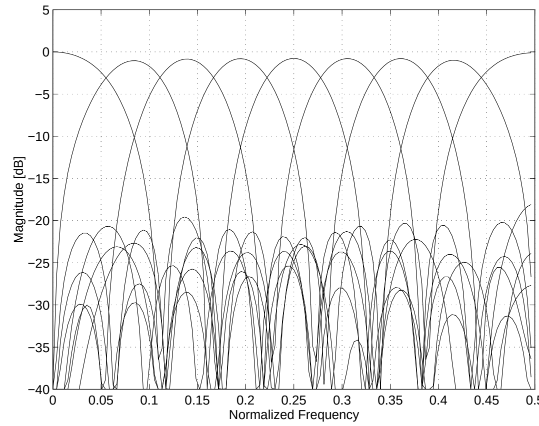

Figure 3. 9-channel paraunitary filter bank with 1-regularity

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