# An Iterative Algorithm for Time-Variant Filtering in the Discrete Gabor Transform Domain

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## Abstract

In this paper, an iterative time-frequency (TF) synthesis/time-varying filtering algorithm in the discrete Gabor transform domain is proposed. A sufficient condition is obtained for the Gabor synthesis and analysis window functions so that the iterative algorithm converges. It is proved that under the condition the algorithm converges to a signal that has its Gabor transform located exactly in a desired domain specified by the user in the TF plane. Under the condition the solution from the first iteration is exactly the least square solution. Our numerical examples show: about 3.5 dB or more SNR gain over the least square solution; about 13dB SNR increase over the SNR without filtering; lower computational complexity.

## 1 Introduction

Traditional linear filtering based upon Fourier transform plays an important role in stationary/narrow band signal extraction/processing. A narrow band signal buried in a wide band noise can be extracted by using a bandpass filter that covers the band the signal occupies. The procedure is simple: take the Fourier transform $\hat{s}(\omega)$ of the noisy signal $s(t)$; mask the spectrum $\hat{s}(\omega)$ by using the bandpass filter $H(\omega)$ as $\hat{s}(\omega)H(\omega)$; take the inverse Fourier transform of the masked spectrum as the filtered signal $\bar{s}(t)$. With this technique a wide band noise will be efficiently rejected while the narrow band signal is still maintained.

The above traditional linear filtering technique will, however, fail when a signal itself is wide band/nonstationary, such as, chirps, speech, and frequency hopped spread spectrum signals. Recently, many new techniques have been developed for exploiting/extracting features of nonstationary signals. The main idea for these techniques is to map one dimensional signals in the time domain into two dimensional signals in the joint TF domains by exploiting the local behavior of nonstationary signals. For convenience, we name these techniques TF transforms.

Usually, TF transforms add redundancy in the TF domain to the signal in the time domain. They spread noise over the whole TF plane and meanwhile contain the signal information in some localized areas as shown in Fig. 1(a)-(c). Therefore, TF transforms usually significantly increase the signal-to-noise ratio in the TF domain. In other words, signals in the TF domain may be easier to be detected than in the time domain alone. With this observation, one might use the following idea for extracting the signal in the time domain analogious to traditional linear filtering: take the TF transform of a noisy signal $s(t)$; mask the TF transform in the TF plane as shown in Fig. 1(d); take the inverse TF transform of the masked TF transform shown in Fig. 1 (d) as $\tilde{s}(t)$.

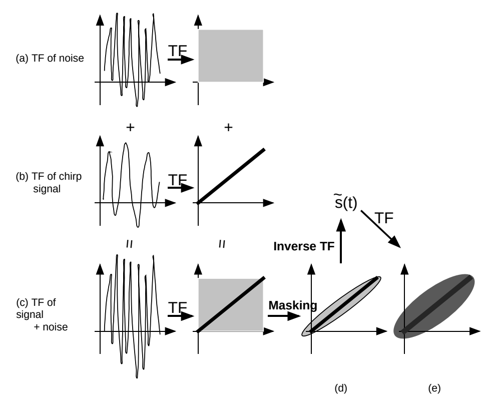


Figure 1: TF transform illustration.

With traditional linear filtering, there is no question about that the Fourier spectrum of the filtered signal $\bar{s}(t)$ has the desired frequency band. This is because that the Fourier transform is a one-to-one and onto mapping. Any signal in the frequency domain corresponds to a unique signal in the time domain. This is, however, no longer true in general for TF transforms. Not every signal in the joint TF domain corresponds to a signal in the time domain due to the fact that TF transforms are redundant and not onto transformations. This implies that the TF transform of the filtered $\tilde{s}(t)$ may not fall in the masked domain as shown in Fig. 1 (d)-(e). With this observation, let us state the general TF synthesis problem (or time-variant fitering). Given a user specified, localized time-frequency domain in the TF plane, find the corresponding time domain waveform. The traditional approach to this problem is the least square solution method, which finds the signal in the time domain that minimizes the squared error between the signal's TF transform and the desired one (see, for example, [1] for ambiguity functions, [3-7]). There are

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two drawbacks to the least square solution method. The first one is that although the error between the TF transform of the solution and the desired one is minimized in the mean squared error sense, the TF transform of the solution is not guaranteed to have the desired time-frequency characteristics. This means the solution may not be the desired one as illustrated later by examples.The second drawback is the computational complexity when signals are fairly long, which is quite often the case in practice. This is because the calculation of the pseudo inverses of the matrices needed for the least square solution method is computationally expensive when their sizes are large.

In this paper, we focus our attentions on one family of TF transforms, which is discrete Gabor transforms. We propose an iterative algorithm for the above time-varying filtering problem. Conditions on the window functions of discrete Gabor transforms are obtained for the convergence of the iterative algorithm. We prove that, under the conditions on the window functions, the discrete Gabor transform of the limit of the time waveforms from the iterative algorithm does have the desired TF domain specified by the user. Furthermore, we prove that, under the conditions, the first iteration of the iterative algorithm is exactly the least square solution. Improvement over the least square solution occurs with more iterations, which can be seen clearly from our numerical examples. From our various numerical examples, about 3.5dB or more signal-to-noise (SNR) ratio gain can be obtained over the least square solution. The SNR with the iterative algorithm increases about 13dB over the original one without filtering.

## 2 An Iterative Algorithm

**Review of discrete Gabor transform**. For more details about discrete Gabor transforms, see [8-14]. Let a signal $s[k]$, a synthesis window function $h[n]$ and an analysis window function $\gamma[n]$ be all periodic with same period $L$, which satisfy the Wexler-Raz identity

$$\sum_{k=0}^{L-1} h[k+mN]W_L^{-nMk}\gamma^*[k] = \delta[m]\delta[n], \qquad (2.1)$$

where $0 \leq m \leq \Delta N - 1, 0 \leq n \leq \Delta M - 1$, $\Delta M$ and $\Delta N$ are the time and the frequency sampling interval lengthes, and $M$ and $N$ are the numbers of sampling points in the time and the frequency domains, respectively, $M \cdot \Delta M = N \cdot \Delta N = L$, $MN \geq L$ (or $\Delta M \Delta N \leq L$). The critical sampling case is when $M \cdot N = \Delta M \cdot \Delta N = L$.

Discrete Gabor transform (DGT) $G$ and its inverse (IDGT) $H$ can be also represented in the following matrix forms. Let

$$\mathbf{C} = (C_{0,0}, C_{0,1}, \cdots, C_{M-1,N-1})^T,$$

$$\mathbf{s} = (s[0], s[1], \cdots, s[L-1])^T.$$

The DGT can be represented by the $MN \times L$ matrix $G_{MN\times L}$ with its $(mN+n)$th row and $k$th column element

$$\gamma^*_{m,n}[k] = \gamma^*[k - m\Delta M]W_L^{-n\Delta Nk}.$$

The IDGT can be represented by the $L \times MN$ matrix $H_{L\times MN}$ with its $k$th row and $(mN+n)$th column element

$$h_{m,n}[k] = h[k - m\Delta M]W_L^{n\Delta Nk}.$$

Thus,

$$\mathbf{C} = G_{MN\times L}\mathbf{s} \quad \text{and} \quad \mathbf{s} = H_{L\times MN}\mathbf{C}. \qquad (2.2)$$

The condition (2.1) implies that

$$H_{L\times MN}G_{MN\times L} = I_{L\times L}, \qquad (2.3)$$

where $I_{L\times L}$ is the $L \times L$ identity matrix.

**Iterative Algorithm in the Gabor domain**: As mentioned in Introduction, the oversampling of the DGT adds redundancy, which is usually prefered for noise reduction applications. From (2.2)-(2.3), one can see that an $L$ dimensional signal $\mathbf{s}$ is transformed into an $MN$ dimensional signal $\mathbf{C}$ and $MN$ is greater than $L$ due to the oversampling. Therefore, only a small set of $MN$ dimensional signals in the TF plane have their corresponding time waveforms with length $L$. Let $D_{MN\times MN}$ denote the mask transform, specifically, a diagonal matrix with diagonal elements either 0 or 1. Let $\mathbf{s}$ be a signal with length $L$ in the time domain. The first step in the time-varying filtering is to mask the TF transform of $\mathbf{s}$

$$\mathbf{C}_1 = D_{MN\times MN}G_{MN\times L}\mathbf{s},$$

where $D_{MN\times MN}$ masks a desired domain in the TF plane. Since the DGT $G_{MN\times L}$ is a redundant transformation, the IDGT of $\mathbf{C}_1$, $H_{L\times MN}\mathbf{C}_1$, may not fall in the mask. In another words, generally,

$$G_{MN\times L}H_{L\times MN}\mathbf{C}_1 \neq D_{MN\times MN}G_{MN\times L}H_{L\times MN}\mathbf{C}_1, \qquad (2.4)$$

where $MN > L$, which is illustrated in Fig. 1(e). Notice that, in the critical sampling case, i.e., $MN = L$, the inequality (2.4) becomes equality. An intuitive method to reduce the difference between the right and the left hand sides of (2.4) is to mask the right hand side of (2.4) again and repeat the procedure, which leads to our iterative algorithm:

$$\begin{aligned} \mathbf{s}_0 &= \mathbf{s}, & (2.5)\\ \mathbf{C}_{l+1} &= D_{MN\times MN}G_{MN\times L}\mathbf{s}_l, & (2.6)\\ \mathbf{s}_{l+1} &= H_{L\times MN}\mathbf{C}_{l+1}, & (2.7)\\ & l = 0, 1, 2, .... \end{aligned}$$

The above iterative algorithm is illustrated in Fig. 2.

The complexity for the iterative algorithm (2.5)-(2.7) is low, which does not need to compute inverses of matrices. By considering the DGT and IDGT, the computational complexity in (2.5)-(2.7) is proportional to the signal length multiplied by the window length, i.e., $LL_W$. Notice that the complexity of directly computing the inverse matrices in the least square solution is proportional to $L^3$. Therefore, when the length of window functions $h$ and $\gamma$ is much shorter than the length of the signal $\mathbf{s}$, the computational complexity in the iterative algorithm (2.5)-(2.7) is much lower than the one for the least square solution.

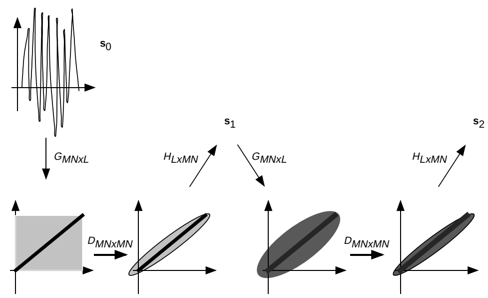


Figure 2: Iterative time-varying filtering algorithm.

The following main results are based on the condition

$$\sum_{l=0}^{\Delta N-1} \gamma^*[lN+k]h[lN+k+m\Delta M]$$

$$= \sum_{l=0}^{\Delta N-1} h^*[lN+k]\gamma[lN+k+m\Delta M], \qquad (2.8)$$

for $k = 0, 1, ..., N-1$ and $m = 0, 1, ..., M-1$.
**Theorem 1**: The product matrix $G_{MN\times L}H_{L\times MN}$ is Hermitian if and only if condition (2.8) for the window functions $h$ and $\gamma$ holds.

There are two trivial cases where the condition (2.8) holds. The first case is the orthogonal-like case: $h[n] = \gamma[n]$ for all integer $n$. The second case is the critical sampling case: $\Delta M = N$. Notice that the continuous Gabor transform is never orthogonal-like unless the window functions are badly localized in the frequency domain. This, however, is not the case for the DGT. The most orthogonal-like solution was studied by Qian et. al. in [8-12]. They showed that it is possible to have the analysis window function $\gamma$ very close to the synthesis window function $h$ when $h$ is truncated Gaussian. The error between $h$ and $\gamma$ is less than $2\times 10^{-6}$ while they are of unit energy, and therefore the error is negligible. We will see numerical results later in the next section.
**Theorem 2**: Under condition (2.8), the DGT of the limit $\bar{\mathbf{s}}$ of the iterative algorithm (2.5)-(2.7) falls in the mask $D_{MN\times MN}$, i.e.,

$$G_{MN\times L}\bar{\mathbf{s}} = D_{MN\times MN}G_{MN\times L}\bar{\mathbf{s}}. \qquad (2.9)$$

With the above result, one might ask whether it violates the known fact that an image of a TF transform of a signal in the TF plane can not be compact support. This is because that a signal can not be time and band limited simutaneously. To answer this question, we first need to know that the above known fact is true for continuous TF transforms. Moreover, the proof of the fact is based upon the marginal properties of TF transforms. It may not be true for discrete TF transforms. In other words, discrete TF transforms may have compact support [11].
**Theorem 3**: Under condition (2.8), the first iteration $\mathbf{s}_1$ of the iterative algorithm (2.5)-(2.7) is equal to the least square solution.

With Theorem 3, one will see in the next section that the iterative algorithm (2.5)-(2.7) improves the least square solution when number of iterations increases, and meanwhile one does not need to compute the inverse matrix.

## 3 Numerical Example

The test signal is a chirp signal, where its time waveform and its Fourier spectrum are shown in Fig. 3:

$$x[n] = \cos(2\pi((n+1)/115)^3), \ \ 0 \le n < 500. \qquad (3.10)$$

The received noisy signal, shown in Fig. 4, is $s(n) = x[n] + \eta[n]$, where $\eta[n]$ is white Gaussian noise. From Figs. 3-4, it can be seen that the traditional Fouier transform filtering may not work in reducing the noise.

The window function length is chosen 256, while the above signal length is 500. The time sampling interval length $\Delta M = 16$ and the frequency sampling interval length $\Delta N = 2$. The window functions we used, their difference, and the absolute values of the differences between the left and right hand sides of condition (2.8) are shown in Fig. 5, respectively. One can see that for this pair the synthesis and the analysis window functions are almost same and condition (2.8) is satisfied numerically. We apply the iterative algorithm (2.5)-(2.7) to extract the signal in (3.10). The signal-to-noise ratio vs. iteration is shown in Fig. 6 as well as the difference between the Gabor transform of the solution and the desired one. As shown in Fig. 6, a 3.5dB gain over the least square solution method (marked by a *) is obtained. The original SNR is $-2.4$ dB while the SNR for the filtered signal is 11.7dB. We also tested another pair of window functions that do not satisfy condition (2.8) and found that the iterative algorithm does not converge.

## 4 Conclusion

In this paper, we proposed a new time-varying filtering algorithm for wideband/nonstationary signals, where traditional filtering in the Fourier domain fails. We obtained a sufficient condition on the convergence of the iterative algorithm. We proved that, under the condition, the limit of the time waveforms from the iterative algorithms has the desired TF characteristics. We also proved that, under the condition, the first iteration is equal to the least square solution. A numerical example was presented to illustrate the theory. As a remark, the new algorithm performs even better when the rate of the oversampling is higher based upon our various numerical examples.

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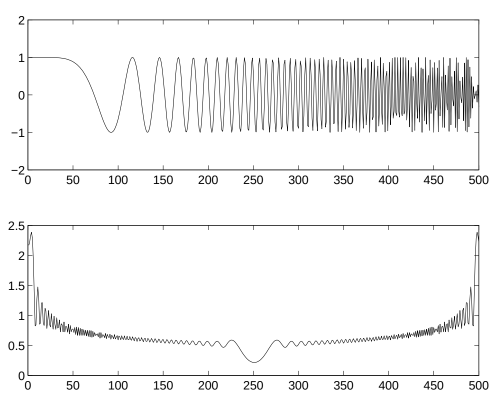

Figure 3: The chirp signal and its Fourier spectrum.

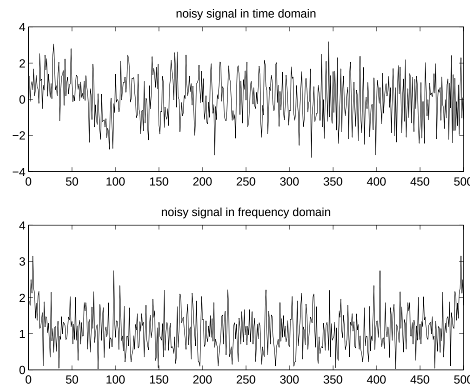


Figure 4: Noisy signal and its Fourier spectrum.

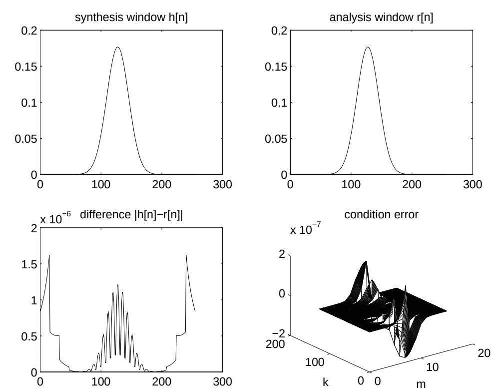


Figure 5: Window functions.

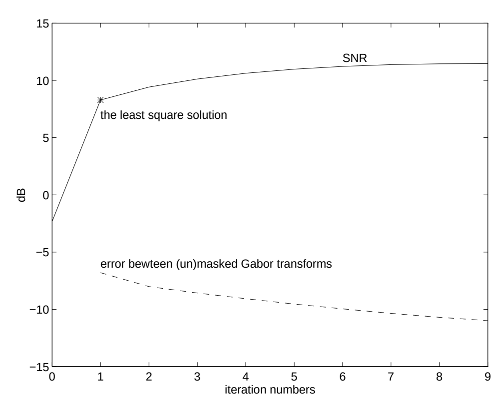


Figure 6: Solid line: SNR vs. iteration steps, where the least square solution is marked by *; Dashed line: The errors between masked and unmasked DGT of the iteration solutions.