Figure 1. The FIR energy compaction filter.

is decimated by M to produce $y(n)$. The optimum FIR energy compaction problem is to maximize the variance

$$\sigma_y^2 = \int_{-\pi}^{\pi} |H(e^{j\omega})|^2 S_{xx}(e^{j\omega}) \frac{d\omega}{2\pi} \quad (1)$$

of $y(n)$ subject to the Nyquist(M) condition [11] on $G(e^{j\omega}) = |H(e^{j\omega})|^2$. Let the impulse response of $G(e^{j\omega})$ be $g(n)$. Then, the Nyquist(M) condition is $g(Mn) = \delta(n)$. Notice that by definition $G(e^{j\omega}) \geq 0$. Define the **compaction gain** as

$$G_{comp}(M, N) = \frac{\sigma_y^2}{\sigma_x^2} = \frac{\int_{-\pi}^{\pi} |H(e^{j\omega})|^2 S_{xx}(e^{j\omega}) \frac{d\omega}{2\pi}}{\int_{-\pi}^{\pi} S_{xx}(e^{j\omega}) \frac{d\omega}{2\pi}} \quad (2)$$

where σ_x^2 is the variance of $x(n)$. The aim therefore is to maximize the compaction gain. As described in [5], the case where $N < M$ and the case where ideal filters are allowed ($N = \infty$) are solved analytically. Our interest is therefore for the case where $M < N < \infty$. Interestingly enough, the window method that we propose involves two stages that can be associated with the above two extreme cases.

3. OVERVIEW AND PROPERTIES

Similar to the IFIR design techniques in conventional filter theory, one can design the compaction filters in multiple stages if M is composite, e.g., $M = M_0 M_1$. This leads to efficiency in both design and implementation. The details of multistage compaction are presented in [4] (see also [5]).

For the two-channel case, the optimal FIR compaction filter can be constructed analytically for some classes of WSS random processes. The method involves representation of positive definite sequences and has connections to other mathematical tools such as line-spectral theory and

Figure 3. The procedure to find $F_L(k)$: $\hat{S}_L(0)$ is maximum among $\{\hat{S}_L(iK)\}$, hence $F_L(0) = M$, $F_L(lK) = 0, l \neq 0$. $\hat{S}_L(1+K)$ is maximum among $\{\hat{S}_L(1+iK)\}$, hence $F_L(1+K) = M$, $F_L(1+lK) = 0, l \neq 1$, and so on.

cess for each $k = 0, \ldots, K-1$, the Fourier series coefficients of the best $f_L(n)$ is determined. The sequence $f_L(n)$ can now be calculated by $f_L(n) = \frac{1}{L}\sum_{k=0}^{L-1} F_L(k) W_L^{-nk}$.

Algorithm for the window method

Assume a window $w(n)$ of the same length as $g(n)$ with nonnegative Fourier transform has been chosen. Let $L = KM > 2N$. Then the algorithm steps are

1. Calculate $\hat{S}_L(k)$, the DFT coefficients of $\hat{r}_L(n), n =$

2. For each $k = 0, \ldots, K-1$, determine the index i_0 for which $\hat{S}_L(k + i_0 K)$ is maximum, and assign $F_L(k + i_0 K) = M$ and $F_L(k + i_l K) = 0,\ l = 1, \ldots, M-1$.
3. Determine $f_L(n)$ and form $g(n) = w(n) f_L(n)$.
4. Spectrally factorize $G(z)$ to find $H(z)$.

If the input is real, the above algorithm can be modified to produce real-coefficient compaction filters. In this case the window $w(n)$ is chosen to be real. Let $P = \frac{K}{2}$ if K is even, and $P = \frac{K-1}{2}$ if it is odd. Then the algorithm for the real process replaces step 2 by the following two steps:

1. For each $k = 0, \ldots, P$, determine the index i_0 for which $\hat{S}_L(k + i_0 K)$ is maximum,
2. If $k + i_0 K = 0$ or $k + i_0 K = \frac{L}{2}$ then set $F_L(k + i_0 K) = M$, else if $k = 0$ or $k = \frac{K}{2}$, then set $F(k + i_0 K) = F(L - k - i_0 K) = \frac{M}{2}$, else, set $F(k + i_0 K) = F(L - k - i_0 K) = M$. Set all the remaining values to zeros.

Optimization of the window

If we fix $f_L(n)$, what is the best window $w(n)$? The objective (1) can be written as

$$\sigma_y^2 = \int_{-\pi}^{\pi} \hat{S}_{xx}(e^{j\omega}) W(e^{j\omega}) \frac{d\omega}{2\pi} \tag{8}$$

where $\hat{S}_{xx}(e^{j\omega})$ is the Fourier transform of $f^*(n) r(n)$ where $f(n)$ is one period of $f_L(n)$ centered at $n = 0$. Let $W(e^{j\omega}) = |A(e^{j\omega})|^2$, where $A(z) = \sum_{n=0}^{N} a(n) z^{-n}$ is the spectral factor of $W(e^{j\omega})$. The only constraint on $A(e^{j\omega})$ is that it has to have unit energy in view of $w(0) = \int_{-\pi}^{\pi} |A(e^{j\omega})|^2 \frac{d\omega}{2\pi} = 1$. Hence, by Rayleigh's principle [3], (8) is maximized if $A(z)$ is the maximal eigenfilter of $\mathbf{P}$. The corresponding compaction gain is the maximum eigenvalue of $\mathbf{P}$.

We have described how to optimize $w(n)$ given $f_L(n)$, and vice versa. It is reasonable to expect that one can iterate and obtain better compaction gains at each stage. We have observed in most examples that two stages of iterations were sufficient to get near-optimal compaction gains. We started with a triangular window and found that $f_L(n)$ did not change after the reoptimization of the window. Notice that, the use of an initial window is not necessary if one is willing to use a window after finding $f_L(n)$. However, in most of the design examples we considered, we have observed that using an initial window with nonnegative Fourier transform (in particular, the triangular window) and then reoptimizing the window resulted in better compaction gains.

Example 1: MA(1) process. Let $N = 5,\ M = 4,\ r(0) = 1,\ r(1) = \rho$, and $r(n) = 0,\ n > 1$. Assume the process is real so that $r(-n) = r(n)$. Let the window be triangular, i.e.,

$$w(n) = \begin{cases} 1 - \frac{|n|}{6}, & n = 0,\ \pm 1,\ \ldots,\ \pm 5 \\ 0, & \text{elsewhere.} \end{cases} \tag{9}$$

The Fourier transform of $\hat{r}(n) = w(n) r(n)$ is $\hat{S}(e^{j\omega}) = 1 + \frac{5}{3}\rho\cos\omega$. Hence, the DFT coefficients $\hat{S}_L(k)$ of $\hat{r}(n)$ in step 1 are $\hat{S}_L(k) = 1 + \frac{5}{3}\rho\cos(\frac{2\pi}{L}k),\quad k = 0, \ldots, L-1$. Now, assume $L = 12 > 10$, so that $K = 3$ and $P = 1$. So we have the following sets to consider in step 2: $\{\hat{S}_L(0), \hat{S}_L(3), \hat{S}_L(6), \hat{S}_L(9)\}, \{\hat{S}_L(1), \hat{S}_L(4), \hat{S}_L(7), \hat{S}_L(10)\}$ which are evaluated as $\{1 + \frac{5}{3}\rho, 1, 1 - \frac{5}{3}\rho, 1\},\quad \{1 + \frac{5\sqrt{3}}{6}\rho, 1 - \frac{5}{6}\rho, 1 - \frac{5\sqrt{3}}{6}\rho, 1 + \frac{5}{6}\rho\}$. First assume $\rho > 0$. The maximum of the first set is $\hat{S}_L(0)$ and the maximum of the second set is $\hat{S}_L(1)$. Hence applying step 3 of the algorithm we have $\{F_L(k),\ k = 0, \ldots, L-1\} = \{4, 4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 4\}$. This determines $f_L(n)$, and $G(z) = \frac{1-\sqrt{3}}{18} z^5 + \frac{1}{6} z^3 + \frac{4}{9} z^2 + \frac{5(1+\sqrt{3})}{18} z + 1 + \frac{5(1+\sqrt{3})}{18} z^{-1} + \frac{4}{9} z^{-2} + \frac{1}{6} z^{-3} + \frac{1-\sqrt{3}}{18} z^{-5}$. The corresponding compaction gain is $1 + \frac{5(1+\sqrt{3})}{9}\rho \simeq 1 + 1.5178\rho$. An optimum compaction filter $H(z)$ is obtained by spectrally factorizing $G(z)$. If $\rho < 0$, it can be verified that the resulting filter will be $H(-z)$ where $H(z)$ is the solution for the previous case.

For comparison, we have also designed an optimum compaction filter using the linear programming technique. The corresponding compaction gain is approximately $1 + 1.6657|\rho|$. This is achieved by using $L = 512$ and a triangular window of order $L - N - 1$. The compaction gain of the window method is only slightly lower. Let us find the improvement we can get by optimizing the window when we fix $f_L(n)$. The compaction gain is the maximum eigenvalue of the 6×6 symmetric Toeplitz matrix with the first row $[1\ f_L(1)\ \rho\ 0\ 0\ 0\ 0]$. This eigenvalue is $1 + 1.8019 f_L(1)|\rho|$. Using $f_L(1)$ from the above calculations, the improved compaction gain is $1 + 1.6410|\rho|$ which is very close to the linear programming compaction gain $1 + 1.6657|\rho|$.

Given this optimal window, can we improve the compaction gain further by reoptimizing $f_L(n)$? In this and all the other design examples we considered, we used the triangular window and then found the optimum $f_L(n)$, and then reoptimized $w(n)$ for $f_L(n)$. Interestingly enough, the reoptimization of $f_L(n)$ did not change it!

Choice of the periodicity L

Increasing L does not necessarily increase the resulting compaction gain. For example using $L = \infty$ which corresponds to using optimum ideal filter $f_L(n)$ for the autocorrelation sequence $\hat{r}_L(n)$ does not result in the best achievable compaction gain using the algorithm. This is true even if no initial window $w(n)$ is used. For the above example, we increased L to 16 and found that the compaction gain decreased! When we used the ideal filter for $f_L(n)$ which corresponds to $L = \infty$, the compaction gain was better than that of the case $L = 16$ but worse than that of the case $L = 12$.

Until this point we assumed that $L > 2N$. If we use a period L that is the smallest multiple of M such that $L \geq 2N$, then we obtain very good compaction gains. This choice can be compactly written as

$$L = M\lceil 2N/M \rceil \tag{10}$$

If $L = 2N$, the sequence $\hat{r}_L(n)$ has the following first period:

$$\{\hat{r}(0), \hat{r}(1), \ldots, \hat{r}(N) + \hat{r}^*(N), \ldots, \hat{r}^*(1)\}. \tag{11}$$

In this case, we have $\hat{r}_L(N) = 2\hat{r}(N)$. This will always be the case if $M = 2$, since $L = 2N$ is a multiple of M.

Connection between the linear programming and window methods

As explained in [5], in the linear programming method, one finds a sequence whose Fourier transform is nonnegative only at a prescribed set of frequencies. To assure the nonnegativity of $G(e^{j\omega})$, one modifies this solution by windowing it. When L is a multiple of M, a periodic sequence $g_L(n)$ in the linear programming method, and a periodic sequence $f_L(n)$ in the window method are found such that they are Nyquist(M) and their Fourier series coefficients are all nonnegative. For $L > 2N$, two problems are not the same because $g_L(n)$ is necessarily zero for some n, while $f_L(n)$ can be nonzero for all n (except $n = kM$, of course). If however $L = 2N$, then the two problems are exactly the same! If windowing is done in the same way in both methods, then we see that the resulting compaction gains should be the same. Hence, one can view the window method as an efficient and noniterative technique to solve a linear programming problem when $L = 2N$. If L is inreased, we saw that the window method does not necessarily yield better gains whereas this is the case for the linear programming method provided the window order is increased as well. However, optimization of the window becomes costly as the order increases. If one uses a fixed triangular window (with a high order) in the linear programming, and if the windows are optimized in the window method, then window method is very close and sometimes superior to the linear programming method as we demonstrate in the following example.

Example 2: Comparison of linear programming and window methods. We have designed compaction filters for an AR(5) input. The psd and the magnitude square of a compaction filter for $(M, N) = (2, 65)$ designed by linear programming are shown in Fig. 4. In Fig. 5(a) we plot for

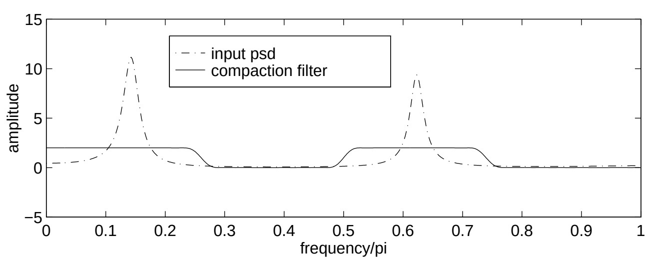

Figure 4. The psd of an AR(5) process, and the magnitude square of an optimal compaction filter designed by linear programming (M = 2, N = 65).

$M = 2$, the compaction gains of both the linear programming and the window method versus the filter order.

The number of frequencies used in the linear programming method is $L = 512$ while the periodicity used in the window method is $L = 2N$. The windows used in the linear programming are triangular windows with order $L - N - 1$. In the window method, the autocorrelation sequence is first windowed by a triangular window of symmetric order N to find $f_L(n)$ and then the window is reoptimized.

From the figure we observe that if the order is high, one has slightly better compaction gains using the window method. This implies that, if one optimizes the window, there is no need to use large number of frequencies in the linear programming method! More importantly, there is no need to use the linear programming technique for high filter

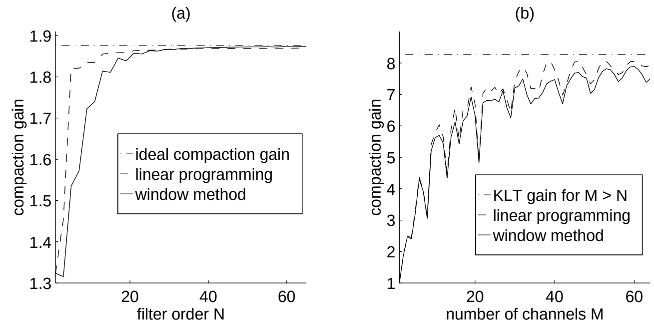

Figure 5. Comparison of the window and linear programming methods. (a) Compaction gain vs. filter order, $M = 2$, (b) Compaction gain vs. number of channels, $N = 65$.

orders. Notice that for high filter orders linear programming method has prohibitively large computational complexity.

In Fig. 5(b), we show the plots of the compaction gains of the two methods versus M for $N = 65$. We observe that the window method performs very close to the linear programming method especially for low values of M. We show the upper bounds on compaction gains in both plots. The upper bound in the first plot is achieved by an ideal compaction filter and that in the second plot is achieved by a maximal eigenfilter [5].

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