FRACTIONAL DISCRETE-TIME LINEAR SYSTEMS

by

Manuel Duarte Ortigueira

INESC/IST, R. Alves Redol, 9, 2º, Lisbon, Portugal

Abstract

In this paper, the class of discrete linear systems is enlarged with the inclusion of the discrete-time fractional linear systems. These are systems described by fractional difference equations and fractional frequency responses. It is shown how to compute the impulse response and transfer function. The theory is supported by the Cauchy integrals that perform projections of the frequency response into or outside the unit circle. The presented formalism is similar to the usually followed

1. INTRODUCTION

Fractals, fractional noises, 1/f noises, fractional differencing became keywords of a lot of published works. This is due to the great importance in practice. In particular, the called 1/f noises are very frequent [3].

In a previous paper [5], we presented an introduction to the fractional continuous-time linear systems. As referred there, those systems may be suitable for modelling several signals and systems found in practice like: fractals, fractional noises, 1/f noises, etc. There are other similar situations where the continuous-time approach is not suitable; for example in hydrology and economics there are time series with fractional characteristics that are not well fitted by the usual ARMA models [2]. The results obtained for continuous-time fractional systems motivate us to consider the discrete-time counterpart [6]. The notions of delay and lag are defined usually only for multiple integer of a given time interval, as shown in the following table.

	z^{-k}	…	z^{-2}	z^{-1}	1	z	z^{2}	…	z^{k}	…
…	$\delta(n-k)$	…	$\delta(n-2)$	$\delta(n-1)$	$\delta(n)$	$\delta(n+1)$	$\delta(n+2)$	…	$\delta(n+k)$	…

Table 1

Essentially and by analogy with the fractional differintegration [5], we would like to generalise that notions to admit fractional delay and lag (delag). In other words we would like to obtain sequences $\delta_\alpha(n)$: bilateral, with $e^{j\omega\alpha}$ as FT, or causal, and anti-causal, having z^α as Z-Transform (ZT). The first case is well known [4,6] and is stated in the following definition of fractional delay and lag (delag):

$$x_{n+\alpha} = \sum_{m=-\infty}^{+\infty} x_m \frac{\sin[\pi(\alpha+n-m)]}{\pi[\alpha+n-m]} \tag{1}$$

where $\alpha\in R$ and $n\in Z$. This equation expresses a relation between two signals x_n and $y_m = x_{n+\alpha}$ defined in the sets Z and { m: m=n+α, n∈Z and α∈R}, respectively. So, we are relating two signals defined over two different time grids, obtained one from the other by a fractional translation. The relation (1) is a convolution of x_n and a $h_\alpha(n)$ given by:

$$h_\alpha(n)=\frac{\sin[\pi(\alpha+n)]}{\pi(\alpha+n)} = \frac{\sin(\pi\alpha)}{\pi\alpha}\cdot\frac{(-1)^n}{1+\frac{n}{\alpha}} \tag{2}$$

which can be considered as the impulse response of a reconstruction filter, $h_\alpha(n)$, such that $x_{n+\alpha} = x_n * h_\alpha(n)$. Remark that

a) This filter is non causal and IIR.

b) We did not introduce any restriction to the delags.

We can write (1) in the form:

$$x_{n+\alpha} = x_n * \delta_\alpha(n) \tag{3}$$

where we represented $h_\alpha(n)$ by $\delta_\alpha(n)$:

$$\delta_\alpha(n) = \begin{cases} \delta(n-\alpha) & \text{if } \alpha \text{ integer} \\ h_\alpha(n) & \text{if } \alpha \text{ non integer} \end{cases} \tag{4}$$

If α is greater than 1, we can put $\alpha=k+\nu$ and we will obtain:

$$\delta_{k+\nu}(n)=\frac{\sin[\pi(\alpha+n+k)]}{\pi(\alpha+n+k)} = \delta_\nu(n+k) \tag{5}$$

Return to equation (1) and apply the FT to both members. We obtain

$$X_\alpha(e^{j\omega}) = e^{j\omega\alpha} X(e^{j\omega}) \tag{6}$$

generalising a well known result. The next steps lead us to try a similar definition concerning the delag in the context of the Z Transform (ZT). Unfortunately, <u>there are no causal nor anti-causal sequences having a ZT equal to z^α ($|z|\neq 1$)</u>. This important fact forces us to use another way: the Cauchy integrals. These are essentially operators that perform a projection of a function defined on the unit circle over the regions outside (causal case) or inside (anti-causal case) the unit circle. Consider a function g(w) defined on the unit circle. Let **L** will be the open unit circle $\{z: z=e^{j\omega}, \omega\in[-\pi,\pi[\}$. This circle decomposes the **C** plane into two regions: the interior $C_i = \{z: |z| < 1\}$ and the exterior $C_e=\{z: |z|>1\}$.

We define the Cauchy problem as consisting in the determination of two functions, G^- and G^+ defined and continuous over $C_i \cup L$ and $C_e \cup L$, respectively, analytic inside their domain, and verifying:

$$G^-(z) + G^+(z) = g(z) \qquad \forall z \in L \qquad (7)$$

We can consider G^- and G^+ as extrapolations of g(z) for the interior and exterior of the unit circle. These are given by:

$$G^+(z) = \frac{1}{2\pi j}\int_L \frac{g(w).w^{-1}}{1-wz^{-1}}\,dw$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{g(e^{j\omega})}{1-e^{j\omega}z^{-1}}\,d\omega \qquad |z|>1 \qquad (8)$$

and

$$G^-(z) = \frac{1}{2\pi j}\int_L \frac{g(w).z.w^{-1}}{w-z}\,dw$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{g(e^{j\omega}).z}{e^{j\omega}-z}\,d\omega \qquad |z|<1 \qquad (9)$$

In other words, we found out how to compute the ZT of the causal part of a given signal, from its Fourier Transform.

2. FRACTIONAL SYSTEMS

The theory we have just presented allows us to treat the Linear Time Invariant Systems characterised by a difference equation like:

$$\sum_{i=0}^{N} a_i\, y(n-i\alpha) = \sum_{k=0}^{M} b_k\, x(n-k\alpha) \qquad (10)$$

with Transfer Function

$$H(e^{j\omega\alpha}) = \frac{\sum_{i=0}^{M} b_i e^{-j\omega i\alpha}}{\sum_{k=0}^{N} a_k e^{-j\omega k\alpha}} \qquad \omega\in[-\pi,\pi[\qquad (11)$$

We will give the name "Fractional Autoregressive Moving Average (FARMA) Systems". We must consider two cases: the general FARMA case and the particular FMA (fractional MA) case. Let h(n) be the inverse Fourier Transform of $H(e^{j\omega})$ (α=1). The FT of x(kαT) is given by:

$$FT[h(k\alpha)] = \begin{cases} \frac{1}{|\alpha|} H(e^{j\omega/\alpha}) & \text{if } |\omega| \le \min(\frac{\pi}{\alpha},\pi) \\ 0 & \text{if } \min(\frac{\pi}{\alpha},\pi)<|\omega|<\pi \end{cases} \qquad (12)$$

For the proof we remark first that

$$\frac{\sin[\pi(k\alpha-m)]}{\pi[k\alpha-m]} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{jw\alpha k}\, e^{-jwm}\,dw \qquad (13)$$

So,

$$FT[h(k\alpha)) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{k=-\infty}^{+\infty} e^{jw\alpha k}\, e^{-j\omega k}\right]\left[\sum_{-\infty}^{+\infty} h(m)e^{-jwm}\right]dw$$

and

$$FT[h(k\alpha)] = \int_{-\pi}^{\pi} \delta(\omega-w\alpha)\, H(w)\,dw \qquad (14)$$

Making wα=η, αdw=dη, and

$$FT[h(k\alpha)] = \frac{1}{|\alpha|}\int_{-\pi/\alpha}^{\pi/\alpha} \delta(\omega-\eta)\, H(\eta/\alpha)\,d\eta \qquad (15)$$

leading immediately to the result stated above. We must remark the limitations imposed in eq. (14) that show that, in general, $H(e^{j\omega\alpha})$ in (11) is not the FT of h(n/α). This happens only when $\alpha\le 1$.

3. COMPUTATION OF THE IMPULSE RESPONSE

The well known results on the geometric series

$$\frac{1}{1-e^{j\omega}z^{-1}} = \sum_{n=0}^{\infty} e^{j\omega} z^{-n} \qquad |z|>1 \qquad (16)$$

is very useful to obtain the impulse responses of the causal systems described by the difference equation (10). In the following we will show how to do it. The simpliest case is the FMA case, because the Transfer Function is given by:

$$H(e^{j\omega\alpha}) = \sum_{m=0}^{M} b_m e^{-jm\omega\alpha} \qquad (17)$$

We only have to use (17) and (16) into (8) to obtain $H^+(z)$. We have:

$$H^+(z) = \sum_{m=0}^{\infty} z^{-n} \sum_{m=0}^{M} b_m \delta_{m\alpha}(n) \qquad (18)$$

with $\delta_{m\alpha}(n)$ given by (2). So, the impulse response is:

$$h(n) = \sum_{m=0}^{M} b_m \delta_{m\alpha}(n) \qquad n\ge 0 \qquad (19)$$

If α is rational, α=p/q, it is not difficult to obtain

$$h(n) = \sum_{k=0}^{\lfloor Mp/q \rfloor}\sum_{i=0}^{q-1} b_{(kq+i)/p}.\delta_{i/q}(n-k) \quad n\ge 0 \ ([1]) \qquad (20)$$

If α=1, q=1 and we obtain the usual response corresponding to a FIR system. It is interesting to remark that, if q≠1, the system is nor a FIR, but a IIR, and its Impulse Response goes to zero with n→∞, but

[1] the symbol ⌊.⌋ means integer part.

hyperbolically, not exponentially. Next we are going to consider the general FARMA case. Without restriction we may assume that the fraction in (11) is proper. The computation of the Impulse Response follows the steps:

a) Consider the function H(w), by substitution of w for z^{α} .

b) The polynomial denominator in H(w) is the indicial polynomial or characteristic pseudo-polynomial. Perform the expansion of H(w) into partial fractions like:

$$F(w)=\frac{1}{(1-a.w^{-1})^{k}} \qquad (21)$$

c) Substitute back z^{α} for w.

d) Compute the Impulse Responses corresponding to each partial fraction;

e) Add the Impulse Responses.

The kernel of the problem is in the partial fraction inversion. We are going to proceed to the inversion of the partial fraction (21) with $w=e^{j\omega\alpha}$. Let us consider the case k=1. The convolution properties can be used in the case of a k>1. Alternatively, we can relate (21) with the (k-1)th derivative of $\frac{1}{1-a.w^{-1}}$ and use the differentiation property of the Z-transform. Now, the problem resumes in the computation of the inverse ZT of:

$$F^{+}(z)=\frac{1}{2\pi j}\int_{L}\frac{1}{1-a.w^{-\alpha}}\frac{w^{-1}}{1-wz^{-1}}dw \qquad |z|>1 \qquad (22)$$

Assume that |a|<1. In this case, we have:

$$\frac{1}{1-a.e^{-j\omega\alpha}}=\sum_{n=0}^{\infty}a^{n}e^{-j\omega n\alpha} \qquad (23)$$

and, using (16):

$$F^{+}(z)=\sum_{n=0}^{\infty}z^{-n}\sum_{k=0}^{\infty}a^{k}\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{j\quad\alpha)\omega}d\omega \qquad |z|>1 \qquad (24)$$

So, the corresponding Impulse Response is:

$$h^{+}(n)=\sum_{k=0}^{\infty}a^{k}.\delta_{-k\alpha}(n) \qquad n\geq 0 \qquad (25)$$

That is the discrete analogue to the Mittag-Leffler function, used in the fractional continuous time linear systems [6]. It is interesting to remark that this case can be obtained from (19) with M=∞ and b_m an exponential sequence. If α is rational, $\alpha=p/q$, put $\nu=1/q$, with $q\neq 1$. In this case we can obtain a different result involving fractional delags less than 1, as in (20). We only have to show the validity of the following formula:

$$\frac{1}{1-a.e^{-j\omega\alpha}}=\frac{\sum_{i=0}^{q-1}a^{i}.e^{-j\omega i\alpha}}{1-a^{q}e^{-j\omega p}} \qquad (26)$$

To obtain it, we only have to sum up the expression in the numerator. On the other hand, |a| <1 allows the use of the geometric series properties to obtain:

$$\frac{1}{1-a.e^{-j\omega\alpha}}=\sum_{i=0}^{q-1}\sum_{k=0}^{+\infty}a^{i+kq}.e^{-j\omega p(k+i\nu)} \qquad (27)$$

Its insertion into (22) leads to:

$$h^{+}(n)=\sum_{i=0}^{q-1}\sum_{k=0}^{+\infty}a^{k.q+i}\,\delta_{-i\alpha}(n-kp) \qquad n\geq 0 \qquad (28)$$

The ZT of this function is not elemental. Remark that this Impulse response goes to zero exponentially and hyperbolically.

4. Stability of fractional systems

The stability of fractional linear time invariant discrete-time systems is easy to study, attending to the way in which we obtain the impulse response. In fact, we transformed the denominators in rational functions of z^{-1}. The properties of the response corresponding to each partial fraction depend on the poles, as in the non fractional case. So, to test the stability of the fractional linear systems we only have to check if the poles are outside of the unit circle. Making $w=z^{\nu}$ in (11) we reduce the stability test to the usual Jury case.

5. Examples

We are going to present some simple examples to illustrate the theory we presented.

1 - FMA case

To illustrate this case we computed the Impulse Response of a FIR filter with Frequency Response represented by a plain line ($\alpha=1$) in figure 1, where we represented Frequency Responses corresponding from outside to inside α=1/3, 1/2, 1, 3/2, 4/3. In figure 2 we show the corresponding Impulse Responses [α=1/3, 1/2, 1 in the upper half picture and α=3/2, 4/3 in the other]. These and other similar results we obtained show that the FMA systems are also FIR.

2 - FAR case

In this example we will present a system with a pair of complex conjugate poles.

$$H(e^{j\omega\alpha})=\frac{1}{1-a.e^{-j\omega\alpha}}+\frac{1}{1-a^{*}.e^{-j\omega\alpha}} \qquad (29)$$

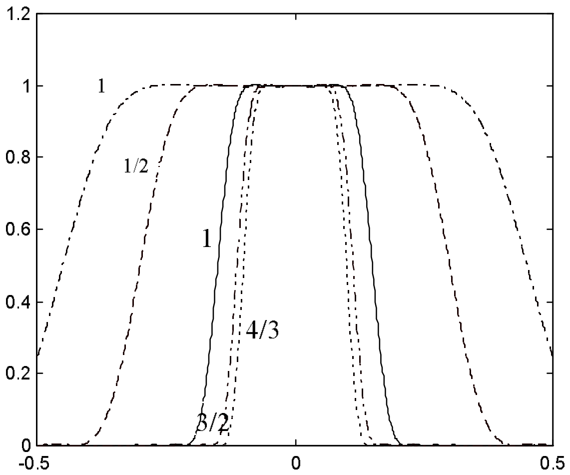

Figure 1

Accordingly to (28) and putting $a=\rho.e^{j\theta}$, the Impulse Response is given by:

$$h^{+}(n) = 2.\sum_{k=0}^{\infty} \rho^{k}.\cos(\theta k)\,.\delta_{-k\alpha}(n) \qquad n\geq 0 \qquad (30)$$

or, if α is rational, by:

$$h^{+}(n) = 2.\sum_{i=0}^{q-1}\sum_{k=0}^{+\infty} \rho^{k.q+i}\cos[(kq+i)\theta]\delta_{-i\alpha}(n-kp) \qquad n\geq 0 \qquad (31)$$

In figure 3, we present an illustration of this function for the previous values of α.

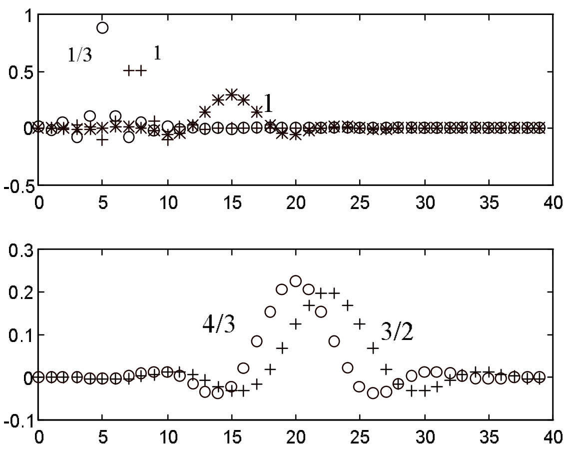

Figure 2

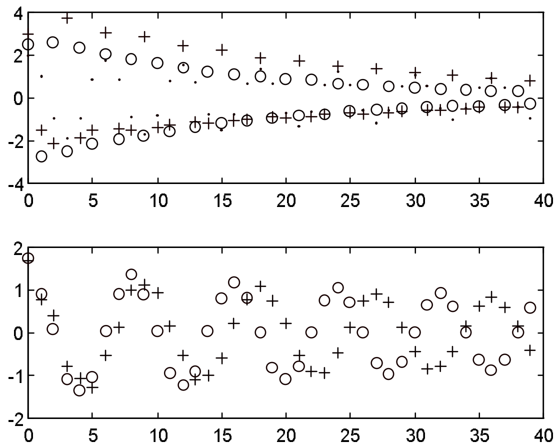

Figure 3

6. CONCLUSIONS

We presented an introduction to the theory of Fractional Discrete-Time Linear Systems. We showed how to compute the Impulse Responses from the Frequency Response. We made a brief study of the stability. There are other interesting problems not focused in this paper that will be subject of study in future publications. In particular, we refer the initial condition problem and the fractional order poles and zeros.

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