ADAPTIVE PERIODIC IIR FILTERS

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ABSTRACT

We consider adaptive periodic IIR filtering and present an extension of the Hyperstable Adaptive Recursive Filter (HARF). We give conditions for convergence of the parameter estimate error, involving passivity of certain operators in the identification loop, identifiability of the system parameters, and persistent excitation (pe). A necessary and sufficient condition for identifiability is given and subject to its satisfaction, input-only conditions guaranteeing pe are given.

1. INTRODUCTION

Linear priodic filters have many important applications. First all multirate filters can be viewed as linear periodic filters, [1]. Second, as noted in [2], modeling of such inherently nonstationary signals as speech, is better accomplished using linear periodic filters than their linear time invariant (LTI) counterparts. By the same token, adaptive multirate filtering is essentially a matter of adaptive linear periodic filtering (ALPF). Similarly, adaptive filtering of speech signals significantly benefits from ALPF. Consequently, this paper studies ALPF.

To this end we adopt an adaptive identification view of ALPF. In particular with $u(k)$ as input and $y(k)$ as output, we assume that

$$y(k) + \sum_{i=1}^{n} a_i(k)y(k-i) = \sum_{j=1}^{m} b_j(k)u(k-j) \qquad (1.1)$$

with

$$a_i(k+N) = a_i(k) \qquad \forall i, k \qquad (1.2)$$

and

$$b_j(k+N) = b_j(k) \qquad \forall j, k. \qquad (1.3)$$

Our goal is to use the knowledge of $u(k)$, $y(k)$, N and n to estimate the periodic sequences $a_i(k)$ and $b_i(k)$.

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Since the basic structure of (1.1) is IIR, in Section 2 we propose a generalization of the well known Hyperstable Adaptive Recursive Filter (HARF), [3], that adaptively estimates the coefficients of IIR LTI systems. Section 3 gives conditions under which this algorithm converges. For parameter convergence, one condition needed is the so called persistent excitation (pe) condition. Section 4 discusses issues connected to the satisfaction of pe. Section 5 is the conclusion.

2. THE ALGORITHM

For convenience in (1.1) write

$$a_i(k) = a_{ij} \qquad \text{if } j = k \bmod N \qquad (2.1)$$

$$b_i(k) = b_{ij}. \qquad \text{if } j = k \bmod N \qquad (2.2)$$

This is possible because of the $N-$periodic nature of the coefficients.

Our task in this section is to adaptively identify a_{ij} and b_{ij} from the input/output (I/O) data. Re-express (1.1) as: for all $0 \le j < N$ and k such that

$$k \bmod N = 0 \qquad (2.3)$$

$$y(k+j) = \sum_{i=1}^{n} a_{ij}y(k+j-i) + \sum_{i=1}^{m} b_{ij}u(k+j-i). \qquad (2.4)$$

Defining for $0 \le j < N$

$$\Psi_j = [a_{1j}, a_{2j}, \ldots, a_{nj}, b_{1j}, b_{2j}, \ldots . b_{mj}]^T, \qquad (2.5)$$

and for all k obeying (2.3)

$$\begin{aligned} X^T(k+j-1) \quad = \quad & [y(k+j-1), \ldots, y(k+j-n), \\ & u(k+j-1), \ldots, u(k+j-m)], \end{aligned}$$

one obtains

$$y(k+j) = \Psi_j^T X(k+j-1). \qquad (2.6)$$

Thus one can view (1.1) as N interlaced LTI systems. Accordingly, the identification process too will be treated in interlaced context, i.e. the estimate of Ψ_j will be updated only at each $(k+j)$-th instant, $k \bmod N = 0$.

Define for all k obeying (2.3) and $0 \leq j < N$

$$\begin{aligned}\hat{\Psi}_j(k+j) &= [\hat{a}_{1j}(k+j), \hat{a}_{2j}(k+j), \ldots, \hat{a}_{nj}(k+j), \\ &\quad \hat{b}_{1j}(k+j), \hat{b}_{2j}(k+j), \ldots . \hat{b}_{mj}(k+j)]^T\end{aligned}$$

as the estimate of Ψ_j at time $k+j$. Next, for each k obeying (2.3) define the *a posteriori* and *a priori* prediction outputs respectively as

$$\underline{\hat{y}}(k+j) = \underline{\hat{X}}(k+j-1)^T \hat{\Psi}_j(k+j) \qquad (2.7)$$

and

$$\hat{y}(k+j) = \hat{X}(k+j-1)^T \hat{\Psi}_j(k+j-1), \qquad (2.8)$$

where

$$\begin{aligned}\hat{X}^T(k+j-1) &= [\hat{y}(k+j-1), \ldots, \hat{y}(k+j-n), \\ &\quad u(k+j-1), \ldots, u(k+j-m)]\end{aligned}$$

and

$$\begin{aligned}\underline{\hat{X}}^T(k+j-1) &= [\underline{\hat{y}}(k+j-1), \ldots, \underline{\hat{y}}(k+j-n), \\ &\quad u(k+j-1), \ldots, u(k+j-m)].\end{aligned}$$

Also define the *a posteriori* and *a priori* prediction errors respectively by

$$\underline{e}(k) = y(k) - \underline{\hat{y}}(k) \qquad (2.9)$$

and

$$e(k) = y(k) - \hat{y}(k). \qquad (2.10)$$

Then the parameter estimate update equations proceed as: for all k obeying (2.3) and $0 \leq j < N$ and $0 \leq l < N$,

$$\hat{\Psi}_j(k+l) = \begin{cases} \hat{\Psi}_j(k+l-1) & \text{if } l \neq j \\ \hat{\Psi}_j(k+l-1) + \delta_{klj} & \text{if } l = j \end{cases} \qquad (2.11)$$

where δ_{klj} is given by

$$\frac{\mu \hat{X}(k+j-1)\left[e(k+j) - \sum_{i=1}^{M} d_{ij}\underline{e}(k+j-i)\right]}{1 + \mu \hat{X}^T(k+j-1)\hat{X}(k+j-1)}. \qquad (2.12)$$

Observe the prediction error is subjected to linear periodic filtering prior to its application in the update kernel.

3. CONVERGENCE

In the LTI case, convergence of the prediction error requires that the system inputs be bounded and that the error system obey a strict passivity condition involving the prediction error filter [4], where strict passivity is defined in Definition 3.1. In addition for exponential convergence of the parameter estimates, one requires a p.e. condition, [4]. Both these facts are also true for the ALPF algorithm given in Section 2.

We begin with a definition of strict passivity.

Definition 3.1: A system with input $\omega(k)$ and output $\upsilon(k)$ is said to be strictly passive if there exist constants K_1 and K_2, with $K_1 > 0$, such that for all bounded $\omega(k)$, initial conditions and k,

$$\sum_{i=0}^{k} \upsilon(i)\omega(i) \geq K_1 \sum_{i=0}^{k} \omega^2(i) + K_2. \qquad (3.1)$$

The constant K_2 reflects the cumulative effect of initial conditions and is zero when the initial conditions are zero. In effect (3.1) states that the average product of the input and the output is positive. It is a well known fact that all strictly passive linear time varying systems are stable.

The passivity condition we need is given by assumption 3.1; the pe condition by assumption 3.2.

Assumption 3.1 *The linear periodic system*

$$\rho(k) - \sum_{i=1}^{n} a_i(k)\rho(k-i) = \eta(k) \qquad (3.2)$$

$$\nu(k) = \sum_{i=1}^{n} d_i(k)\rho(k-i) + \rho(k) \qquad (3.3)$$

is strictly passive, where for every integer k

$$d_i(kN+j) = d_{ij} \qquad \forall\ 0 \leq j < N. \qquad (3.4)$$

The pe assumption in its turn is as follows.

Assumption 3.2 *For each k define the $nN \times N$ matrix Γ_k by*

$$\begin{bmatrix} X(kN-1) & 0 & \cdots & 0 \\ 0 & X(kN) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X(kN+N-2) \end{bmatrix}. \qquad (3.5)$$

Then $\exists\ \delta_1,\ \delta_2 > 0$ and M_2 such that for all k

$$\delta_1 I \leq \sum_{i=k}^{k+M_2} \Gamma_i \Gamma_i^T \leq \delta_2 I. \qquad (3.6)$$

The first Theorem deals with prediction error convergence without pe.

Theorem 3.1 *Under Assumption 3.1, and bounded $u(k)$*

$$\lim_{k\to\infty} \underline{e}(k) = 0. \qquad (3.7)$$

We now turn to exponential convergence of the parameter estimates.

Theorem 3.2 *Suppose (1.1,2.1,2.2) is stable and Assumption 3.1 and 3.2 hold. Then* $\hat{\Psi}(k) - \Psi(k)$ *converges exponentially to zero.*

4. IDENTIFIABILITY AND PERSISTENT EXCITATION

In this section we conduct a deeper study of the pe condition in Assumption 3.2. Observe this condition is phrased in terms of both the input and the output of the systems in question. Since the outputs come from unknown systems, these conditions are of limited practical value. It is important then to derive conditions that depend only on the system inputs and yet force Assumption 3.1 to be satisfied. Such input-only p.e. conditions are what this Section seeks.

In broad terms the pe condition requires the input to be such that the resulting I/O relationship is satisfied by a unique set of parameters. There may be systems for which no such inputs exist. For example an LTI system admitting a pole-zero cancellation can be represented by multiple parameter combinations. Such a situation is termed as lack of *Identifiability.* To ensure input selection that guarantees pe, we must first characterize identifiability for the system in (1.1). This is done in Section 4.1. Subject to this identifiability condition, Section 4.2 provides conditions on $u(k)$ that guarantee pe.

4.1. Identifiability

Define for all integer k

$$Y_k = [y(kN), y(kN+1), \cdots, y(kN+N-1)]^T$$

$$U_k = [u(kN), u(kN+1), \cdots, u(kN+N-1)]^T$$

and the forward shift operator q such that

$$qY_k = Y_{k+1} \qquad \text{and} \qquad qU_k = U_{k+1}. \tag{4.1}$$

Define also

$$L = \lceil \tfrac{n}{N} \rceil, \tag{4.2}$$

where $\lceil a \rceil$ denotes the smallest integer greater than or equal to a. Observe that (1.1) can be expressed as

$$\sum_{i=0}^{L} A_i Y_{k+i} = \sum_{i=0}^{L} B_i U_{k+i}, \tag{4.3}$$

where each A_i and B_i is NxN and moreover obeys the following structure: with

$$A = [A_0, \cdots, A_L], \tag{4.4}$$

$$B = [B_0, \cdots, B_L], \tag{4.5}$$

the ijth element of A obeys

$$A_{ij} = \begin{cases} 0 & \forall L > j \text{ or } \ j > i+n \\ 1 & \forall j = i+n \end{cases} \tag{4.6}$$

and that of B obeys

$$B_{ij} = 0 \qquad\qquad \forall i > j \text{ or } \ j \geq i+n. \tag{4.7}$$

The LTI, Multiple Input Multiple Output (MIMO) system (4.3) is called the lifted version of the linear periodic system. Using (4.1) one can rewrite (4.3) as

$$A(q)Y_k = B(q)U_k, \tag{4.8}$$

where the NxN polynomial matrices are defined as

$$\begin{aligned} A(q) &= A_0 + qA_1 + \cdots + q^L A_L, & (4.9)\\ B(q) &= B_0 + qB_1 + \cdots + q^L B_L. & (4.10) \end{aligned}$$

Observe A has rank N. Thus, $A(q)$ is invertible (see [5]). Thus the lifted transfer function written in left factor form is

$$A^{-1}(q)B(q). \tag{4.11}$$

We next need the following facts.

Fact 4.1: A polynomial matrix $R(q)$ is a *left common factor* of $A(q)$ and $B(q)$ if there exist polynomial matrices $\bar{A}(q)$ and $\bar{B}(q)$ such that

$$\begin{aligned} A(q) &= R(q)\bar{A}(q)\\ B(q) &= R(q)\bar{B}(q). \end{aligned}$$

Fact 4.2: $A(q)$ and $B(q)$ are said to be *left coprime* if all their left common factors are unimodular, i.e. have a constant, non-zero determinant.

Then we have the following result.

Theorem 4.1 *The system (1.1,2.1,2.2) is identifiable iff* $A(q)$ *and* $B(q)$ *are left coprime.*

4.2. Persistent Escitation

Subject to the coprimeness condition in Theorem 4.1, we now give conditions on the input that guarantee pe. To place these results in context we first comment on what happens in the LTI case. Should the $a_i(k)$ and $b_i(k)$ in (1.1) be constant, then, [4], the following is sufficient for the pe condition for HARF: There exist M_1, δ_3, δ_4 such that for all k

$$\delta_3 I \leq \sum_{i=k}^{k+M_1} \mathcal{U}_{2n}(i)\mathcal{U}_{2n}^T(i) \leq \delta_4 I$$

where

$$\mathcal{U}_j(k) = [u(i), u(i+1), \cdots, u(i+j-1)]^T.$$

Should $u(k)$ be a linear combination of sinusoids, then this requires at least n distinct frequency components. This can be reconciled with intuition by noting that in the LTI case of (1.1), there are 2n parameters to be identified. Since each frequency components carries two pieces of information, amplitude and phase, n frequency components should suffice to identify 2n parameters. In practice due to potential aliasing problems one can say n generic frequencies are enough.

In the N-periodic case there are 2nN parameters in all. The foregoing arguments suggest nN input frequencies would be needed. Consider, however the following result, which indicates the contrary.

Theorem 4.2 *Suppose (1.1,2.1,2.2) is identifiable and stable. Define*

$$v = n(N+1) + N - 1.$$

Then Assumption 3.2 holds if there exist M_1, δ_3, δ_4 such that for all k

$$\delta_3 I \leq \sum_{i=k}^{k+M_1} \mathcal{U}_v(iN)\mathcal{U}_v^T(iN) \leq de_4 I.$$

Observe, for $N \geq 1$,

$$\frac{v}{2} \leq nN,$$

with equality only with $N = 1$. Thus in fact for non-trivial periodic systems, a less stringent requirement has been placed. This is evidently a reflection on the ability of a periodic system to generate additional frequencies internal to the system. This excitation enhancement capability is further demonstrated by the next result, which considers inputs of the form

$$u(k) = \sum_{i=1}^{\alpha} C_i e^{j\omega_i k}, \tag{4.12}$$

for complex C_i, real ω_i and integer α. We say that this input has α spectral lines. Observe that a L-frequency real input has 2L spectral lines. Then Theorem 4.3 below shows that the input needs only *2n generic spectral line combinations*, which is also the requirement for a LTI system with 2n unknown parameters. Thus, generically the periodic system generates enough internal excitation to take care of the additional unknown parameters periodicity creates.

Theorem 4.3 *Suppose (1.1,2.1,2.2) is identifiable and stable. Then inputs of the form in (4.12) suffice to force the satisfaction of Assumption 3.2, for $\alpha = 2n$ and all $\omega_1, \cdots, \omega_{2n}$ save those for which $\Omega = [\omega_1, \cdots, \omega_{2n}]^T$ lies on a set of measure zero in R^{2n}.*

5. CONCLUSION

This paper gives an extension of HARF to the linear periodic setting. The extrension involves an interlaced update scheme. Exponential asymptotic stability has been demonstrated under a strict passivity condition coupled and a persistent excitation (p.e.) condition.

We have given a necessary condition for a linear periodic system to be identifiable, and subject to its satisfactions given two input-only conditions guaranteeing p.e.. In the case when $2nN$ unknown constants define the input-output behavior, the first of these ensures p.e. when the input comprises a linear combination of $v < Nn$ distinct frequency components.

The second condition shows that generic n-frequency sinusoids suffice for p.e. . Both these conditions reflect the ability of linear periodic systems to enhance the excitation injected at their inputs.

One important open issue concerns the satisfaction of the passivity condition on a combination of the periodic error filter and the unknown system parameters. For the LTI case, this problem has been solved in [6]. Whether [6] extends to this setting is a subject of current investigation.

6. REFERENCES

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