NOISE CONSTRAINED LMS ALGORITHM

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ABSTRACT

In many identification and tracking problems, an accurate estimate of the measurement noise variance is available. A partially adaptive LMS-type algorithm is developed which can exploit this information while maintaining the simplicity and robustness of LMS. This noise constrained LMS (NCLMS) algorithm is a type of variable step-size LMS algorithm, which is derived by adding constraints to the mean-square error optimization. The convergence and steady-state performance are analyzed. Both the theoretical results and simulations show that NCLMS can dramatically outperform LMS, RLS and other variable step-size LMS algorithms in a sufficiently noisy environment.

1. INTRODUCTION

There continues to be a need for low complexity robust algorithms for acquiring and tracking rapidly varying linear systems/channels. For example, in wireless communications, transmission bandwidths typical of time division multiple access (TDMA) produce significant intersymbol interference (ISI) as well as severe fading; this combination of ISI and fading requires channel estimation algorithms which can track rapid time variations and recover from deep fades. There are a range of techniques available for trained (and decision directed) identification and tracking of linear FIR channels with additive white Gaussian noise (AWGN), which we broadly group into two classes: adaptive and model-based [1]. The adaptive algorithms do not explicitly use a model for the channel coefficients or noise and include least mean squares (LMS), recursive least squares (RLS), etc.. The model-based algorithms use various type of models for the channel coefficients (e.g., random walk, autoregressive, or constant) and noise, where the model parameters are either known or jointly estimated with the channel, and include the Kalman filter.We note that many adaptive algorithms can be interpreted in a model based framework with data-dependent choice of model parameters [2]. Also, some adaptive algorithms implicitly use model parameters to set the algorithm parameters, e.g., step-size and forgetting factors require partial knowledge of input statistics to guarantee stable behavior and also noise statistics to guarantee a certain misadjustment/tracking performance. Clearly, if partial knowledge of the channel is available we should try to use it to improve performance, provided it doesn't increase complexity and/or decrease robustness unduly.

Here we propose LMS-related algorithms for trained (or decision directed) identification and tracking of FIR AWGN channels which exploit assumed knowledge of the channel noise variance (but not the channel coefficients). In fact, accurate estimates of the noise variance are known in many communications applications where physical modelling and/or measurement of channel models along with AGC (automatic gain control) is employed [3]. Although RLS-related algorithms can also be developed, the algorithm we propose retains the simplicity and robustness of the usual LMS algorithm, and its behavior can be examined analytically.

In this paper we show how to incorporate knowledge of the channel noise variance by formulating a constrained optimization problem. A (fixed-gain) Robbins-Munro algorithm is developed for solving this problem. This algorithm, which we call noise constrained LMS (NCLMS), is a type of variable step-size LMS (VSLMS) algorithm which has about the same complexity as LMS. We analyze the steady-state behavior of NCLMS by obtaining an asymptotic expansion for the misadjustment, and show explicitly how knowledge of the noise variance is used in NCLMS to get a significant increase in convergence rate for fixed misadjustment over LMS and other VSLMS algorithms (which do not use knowledge of the noise variance). Extensive simulations are conducted in both stationary and non-

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stationary environments to test NCLMS against other algorithms. The results show that in a sufficiently noisy environment (and with sufficient knowledge of the noise variance) NCLMS can approach or even exceed RLS performance.

2. THE NCLMS ALGORITHM

For simplicity, we shall assume that the channel noise variance is known exactly (the results can be generalized to the case where the variance is known within some tolerance).To develop the NCLMS algorithm we consider the time-invariant channel model

$$y_k = \sum_{i=0}^{L} c_i x_{k-i} + n_k = \underline{c}^T \underline{x}(k) + n_k, \; k = 0, 1, \ldots$$

where x_k is a stationary (input) process with mean 0 and variance σ_x^2, n_k is a stationary (noise) process with mean 0 and variance σ_n^2 , and $\{x_k\}$, $\{n_k\}$ are uncorrelated. Let $R = E\{\underline{x}(k)\underline{x}(k)^T\}$ and $\underline{p} = E\{\underline{x}(k)y_k\}$. Then minimizing the mean square error (MSE)

$$\mathcal{E}(\underline{w}) = E\{e_k^2\} = E\{(y_k - \underline{w}^T \underline{x}(k))^2\}$$

over $\underline{w}$ gives the optimal weight value $\underline{w} = \underline{c}$. Note that although R, $\underline{p}$ do not depend on the channel noise variance σ_n^2, this does not mean that a (partially) adaptive algorithm for estimating $\underline{c}$ cannot exploit knowledge of σ_n^2.

Now consider the following constrained minimization problem which incorporates knowledge of σ_n^2:

$$Minimize\ \mathcal{E}(\underline{w})\ over\ \underline{w}\ subject\ to\ \mathcal{E}(\underline{w}) = \sigma_n^2$$

The Lagrangian for this problem is

$$\mathcal{E}_1(\underline{w}, \lambda) = \mathcal{E}(\underline{w}) + \lambda(\mathcal{E}(\underline{w}) - \sigma_n^2)$$

The critical values of $\mathcal{E}_1(\underline{w}, \lambda)$ are $\underline{w} = \underline{c}$, λ arbitrary. Note that although there are no spurious critical $\underline{w}$, the fact that there is no unique (or even constrained) critical λ will present problems for an iterative/adaptive algorithm. To correct this, we add an additional penalty term $\gamma\lambda^2$ ($\gamma > 0$) to $\mathcal{E}_1(\underline{w}, \lambda)$ to get the augmented Lagrangian

$$\mathcal{E}_2(\underline{w}, \lambda) = \mathcal{E}(\underline{w}) + \lambda(\mathcal{E}(\underline{w}) - \sigma_n^2) + \gamma\lambda^2$$

The unique critical value of $\mathcal{E}_2(\underline{w}, \lambda)$ is $\underline{w} = \underline{c}$, $\lambda = 0$.

To solve for the critical value of $\mathcal{E}_2(\underline{w}, \lambda)$ and hence the channel $\underline{c}$ based on a training sequence $\underline{x}(k)$, y_k, $k =$ 0, 1, ..., we use a fixed gain Robbins-Munro algorithm

$$\begin{aligned} \underline{w}(k+1) &= \underline{w}(k) + \alpha_k e_k \underline{x}(k) \\ \lambda_{k+1} &= \lambda_k + \beta(\frac{1}{2}(e_k^2 - \sigma_n^2)) - \lambda_k) \\ \alpha_k &= \alpha(1 + \gamma\lambda_k) \end{aligned} \tag{1}$$

where α, $\beta > 0$. We call Eq. (1) the *noise constrained LMS* (NCLMS) algorithm. In order to guarantee stability of NCLMS we truncate the step-size α_k to the closed interval $[\underline{\alpha}, \overline{\alpha}]$. It can be shown that NCLMS is stable, i.e., $\underline{w}(k)$ and λ_k are mean-square bounded if $0 \leq \underline{\alpha} \leq \overline{\alpha} < \frac{2}{3Tr\{R\}}$ and $\beta < 1$ (assuming $\underline{x}(k)$, n_k are white Gaussian sequences).

It is seen that NCLMS is in fact a type of variable step size LMS (VSLMS) algorithm. If we set the noise variance σ_n^2 to zero in Eq. (1) we get another VSLMS algorithm

$$\begin{aligned} \underline{w}(k+1) &= \underline{w}(k) + \alpha_k e_k \underline{x}(k) \\ \lambda_{k+1} &= \lambda_k + \beta(\frac{1}{2}e_k^2 - \lambda_k) \\ \alpha_k &= \alpha(1 + \gamma\lambda_k) \end{aligned} \tag{2}$$

We call Eq. (2) the *zero-noise constrained LMS* (ZNCLMS) algorithm. This is a similar algorithm to the one derived [4] and can be shown to have similar properties.

To compare LMS (step-size α), NCLMS and ZNCLMS in a stationary environment, we shall use the following respective asymptotic expansions for their misadjustment valid for $\alpha Tr\{R\} \ll 1$ and derived under approximations similar to [4]:

$$\mathcal{M}_L \approx \frac{1}{2}\alpha TrR \tag{3}$$

$$\mathcal{M}_N \approx \frac{1}{2}\alpha TrR\left(1 + \frac{\gamma^2\beta}{2(2-\beta)}\sigma_n^4\right) \tag{4}$$

$$\begin{aligned} \mathcal{M}_Z \approx\ & \frac{1}{2}\alpha TrR\left(1 + \left(\frac{\gamma}{2}\sigma_n^2 + \frac{\gamma^2}{2(2-\beta)}(1+\frac{\beta}{2})\sigma_n^4\right)\right. \\ & \left.\cdot (1 + \frac{\gamma}{2}\sigma_n^2)^{-1}\right) \end{aligned} \tag{5}$$

A reasonable procedure to select parameters for the purpose of comparing algorithms is to choose α for a specified LMS misadjustment $\mathcal{M}_L = M$, and then choose β, γ in NCLMS and ZNCLMS to maximize the convergence rate while maintaining NCLMS and ZNCLMS misadjustment $\mathcal{M}_N, \mathcal{M}_Z \leq (1+\delta)M$ where δ is some small number, say $\delta = 0.1$ (for fair comparisons the value of α in LMS can then be increased to also give a misadjustment $\mathcal{M}_L = (1+\delta)M$). Now to compare convergence rates we focus on the case where the algorithms are far from steady-state, i.e., $E\{e_k^2\} \gg \sigma_n^2$ and compare their largest time constants (near steady-state they have approximately the same time constant). We see that the time constant for NCLMS and ZNCLMS are both given by

$$\tau \approx \frac{1}{\alpha\gamma E\{e_k^2\}\lambda}$$

However, from Eq. (4) we have that for NCLMS γ can be increased and β decreased without changing the misadjustment, while for ZNCLMS γ can only be increased to a point without increasing the misadjustment. This allows the convergence rate of NCLMS to be dramatically increased over ZNCLMS (and LMS). This demonstrates analytically (within the context of the approximations employed) how the NCLMS algorithm exploits knowledge of the noise variance.

2.1. Examples

First, LMS, RLS and NCLMS are compared in a stationary environment. The channel taps are

$$\underline{c} = [.227, .460, .688, .460, .227]^T$$

($L = 5$) and the other parameters are $R = I$, $\sigma_n^2 = 0.01$ and $\delta = 0.1$. We choose $\alpha = \frac{4}{100TrR}$ for LMS. For NCLMS, $\alpha = \frac{4}{110TrR}$, $\overline{\alpha} = \frac{2}{3TrR}$, $\underline{\alpha} = \frac{2}{3000TrR}$, $\beta = 0.01$ and γ is chosen such that $\delta = 0.1$. For RLS, $R^{-1}(0) = 250I$, $\lambda = 0.992$. With these setting the theoretical misadjustments of all the algorithms are 0.02. Fig. 1 shows the learning curves averaged over 100 trials. The experimental misadjustments of LMS, RLS and NCLMS are 0.0236, 0.0238 and 0.0239 respectively, which is in close agreement with the theory. NCLMS converges at about the same rate as RLS even when RLS is initialized with a favourable value to ensures its stability.

Next, LMS, RLS and NCLMS are compared in nonstationary environments. Fig. 2 shows the third tap of the filter tracking a random walk model $\underline{c}(k) = \underline{c}(k-1) + \underline{v}(k)(L = 5)$, where $\underline{v}(k)$ is white Gaussian noise with zero mean and covariance $0.01I$. Fig. 3 shows the third tap tracking a deterministic phase shift model $\underline{c}(k) = \underline{c}\exp(j2\pi k/1000)$ ($\underline{c}$ as above). NCLMS tracks these models much more accurately than LMS and RLS (which have about the same tracking behavior).

Finally, NCLMS and ZNCLMS are compared with the same parameters (as above) except $\beta = 0.95$ for ZNCLMS (and γ is chosen such that $\delta = 0.1$). The results are in Fig. 4 and Fig. 5. Clearly, NCLMS learns faster and tracks better than ZNCLMS.

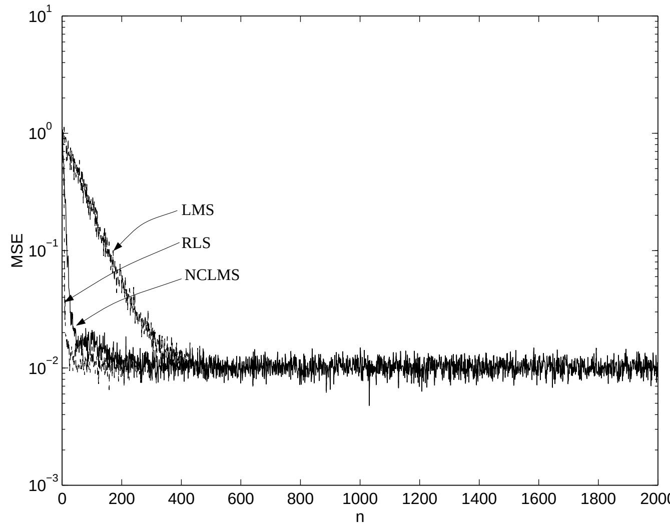

Figure 1: Comparison of LMS, RLS and NCLMS in stationary environment.

3. REFERENCES

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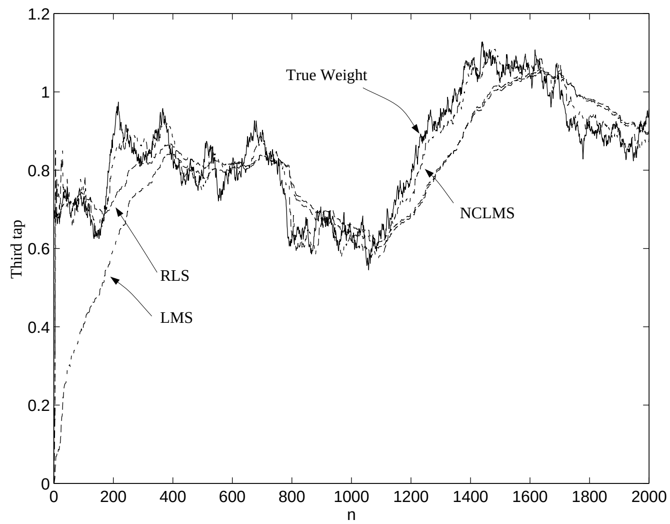

Figure 2: Tracking of LMS, RLS and NCLMS with a random walk model.

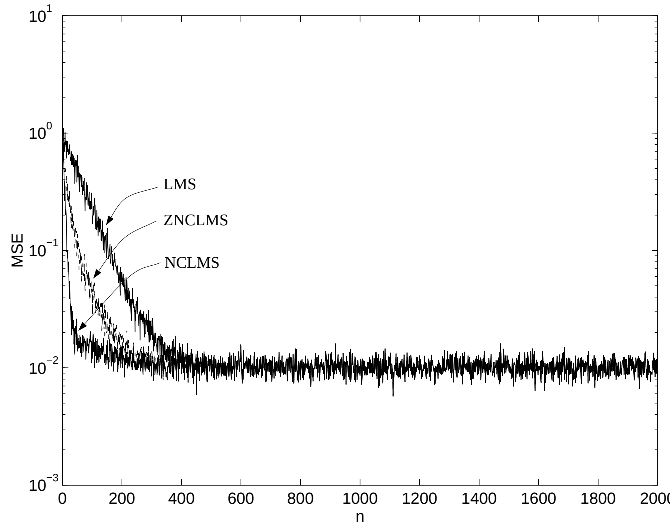

Figure 4: Comparison of LMS, NCLMS and ZNCLMS in stationary environment.

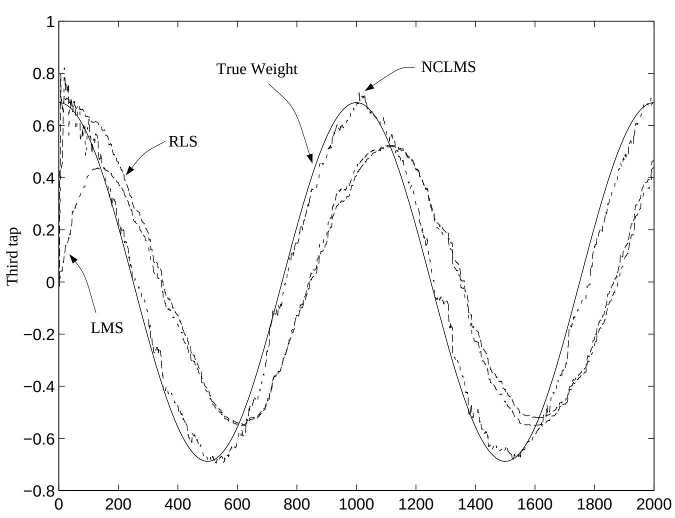

Figure 3: Tracking of LMS, RLS and NCLMS with a phase shift (deterministic) model. Only the real part is shown here.

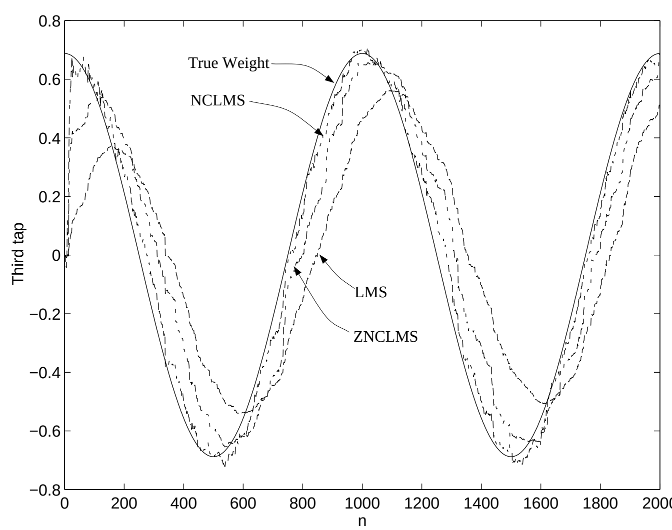

Figure 5: Tracking of LMS, ZNCLMS and NCLMS with a phase shift (deterministic) model. Only the real part is shown here.