

PARAMETRIZATION OF DISCRETE FINITE BIORTHOGONAL WAVELETS WITH LINEAR PHASE

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ABSTRACT

We present a parametrization of discrete finite biorthogonal wavelets with linear phase. Our approach is similar to Zou and Tewfik's for orthogonal wavelets in the way that we utilize a lattice factorization of polyphase matrices of two-channel PR filter banks. However in the biorthogonal case we are faced with the additional possibility of having length differences between the low- and high-pass filters. Our solution to this problem is the introduction of a set of initial polyphase matrices for the lattice product in order to receive the possibility of choosing a certain length difference between the two corresponding filters. This modification of the original lattice product enables us to generate a larger class of discrete wavelets in a systematic way.

1. INTRODUCTION

In many applications of the wavelet transform it is not clear, which wavelet to use. Thus it is often necessary to test performance with a large number of different wavelets. An easy way to systematically generate these wavelets can be very useful in those cases. For orthogonal wavelets there are some parametrizations [1, 2] but it is well known that orthogonal wavelets cannot yield linear phase. However linear phase is a feature that is desired or to be taken in consideration in many applications.

We show in this paper that the parametrization concept from Zou and Tewfik [1] for orthogonal wavelets can be transferred to a parametrization for linear phase discrete finite biorthogonal wavelets. We achieve that by utilizing a lattice factorization likewise, but now it is factorization of linear phase PR filter banks so that we are faced by additional problems. In particular this involves the problem of generating discrete scaling functions and mother wavelets which differ in length.

Instead of two discrete time FIR filters as in the orthogonal case we now have to design four discrete time FIR filters $H(z)$, $G(z)$, $\tilde{H}(z)$ and $\tilde{G}(z)$, the scaling and wavelet filters and its duals. These four filters have to form a 2-channel

perfect reconstruction FIR filter bank and thus $\tilde{H}(z) = G(-z)$, $\tilde{G}(z) = -H(-z)$. Further restrictions apply when these filters should not only lead to discrete wavelets as defined for example by Rioul [3] but to continuous wavelets.

We will first summarize important essentials about biorthogonal wavelets and will then review and extend factorizations of 2-channel perfect reconstruction filter banks with linear phase for two different cases, depending on even or odd length of the filters.

2. BIORTHOGONAL WAVELETS

Discrete biorthogonal scaling functions and wavelets are nothing else than the filters of a 2-channel perfect reconstruction filter bank [3]. Discrete scaling functions correspond to the low-pass FIR filters $H(z)$, $\tilde{H}(z)$ and discrete wavelets correspond to the high-pass FIR filters $G(z)$, $\tilde{G}(z)$. In many applications the discrete point of view is completely sufficient since often only a few iterations of the transform are calculated. However when the aim is on continuous wavelets additional constraints apply to the filters. Essentially one has to assure that the infinite products

$$\begin{aligned}\Phi(e^{-j\omega}) &= (2\pi)^{-1/2} \prod_{k=1}^{\infty} H(e^{-j2^{-k}\omega}) \\ \tilde{\Phi}(e^{-j\omega}) &= (2\pi)^{-1/2} \prod_{k=1}^{\infty} \tilde{H}(e^{-j2^{-k}\omega})\end{aligned}$$

converge to continuous functions and that the functions Ψ , $\tilde{\Psi}$, defined by

$$\begin{aligned}\Psi(e^{-j\omega}) &= e^{-j\frac{\omega}{2}} \overline{\tilde{H}(e^{-j(\frac{\omega}{2}+\pi)})} \Phi(e^{-j\frac{\omega}{2}}) \\ \tilde{\Psi}(e^{-j\omega}) &= e^{-j\frac{\omega}{2}} \overline{H(e^{-j(\frac{\omega}{2}+\pi)})} \tilde{\Phi}(e^{-j\frac{\omega}{2}}),\end{aligned}$$

constitute two dual Riesz bases. Different formulations of necessary and sufficient conditions are available. See for example [4, 3] for further details. As we concentrate on discrete wavelets we do not have to take most of these conditions into account. The only conditions we need are that $H(z)$ is a low-pass, i.e. $H(-1) = 0$ and $H(1) \neq 0$, and that $G(z)$ is a high-pass, i.e. $G(1) = 0$ and $G(-1) \neq 0$.

3. PARAMETRIZATION OF DISCRETE LINEAR PHASE WAVELETS

As mentioned before generating discrete finite biorthogonal wavelets means generating 2-channel perfect reconstruction FIR filter banks. At least for linear phase solutions this leads to lattice factorizations established by [5, 6].

One has to distinguish between two cases:

1. $H(z)$ and $G(z)$ have both even length, their lengths differ by a multiple of 4, $H(z)$ is symmetric, $G(z)$ is antisymmetric or vice versa.
2. $H(z)$ and $G(z)$ have both odd length, their lengths differ by a multiple of 2 but not by a multiple of 4, both are symmetric.

Call them type 1 and type 2, respectively. As in [1] we start by describing the filter bank by its polyphase matrix $H_p(z)$:

$$\begin{bmatrix} H(z) \\ G(z) \end{bmatrix} = H_p(z^2) \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix} \quad (1)$$

In [5, 6] the authors develop representations of $H_p(z)$ in a product form to achieve better filter performance in terms of implementation. Thus they assume that a certain polyphase matrix is given and present algorithms to calculate its factorization in order to achieve an optimal implementation finally.

Our aim is different in this work. We are interested in generating as many perfect reconstruction polyphase matrices with linear phase FIR filters as possible. For this purpose we will use the same product representation but adapt it slightly.

3.1. The Even-length Case

When both, the analysis and the synthesis filters in a PR filter bank should be FIR, $H_p(z)$ not only has to be invertible but its inverse has to consist of finite polynomials. This can be assured by the condition that the determinant of $H_p(z)$ must be a monomial.

In the even-length case a factorization which guarantees this condition is [5, 6]

$$H_p(z) = AS_L\Lambda(z)S_{L-1}\Lambda(z)\cdots S_1\Lambda(z)S_0, \quad (2)$$

$$\Lambda(z) = \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \text{ and } S_i = \begin{bmatrix} \cos \theta_i & \sin \theta_i \\ \sin \theta_i & \cos \theta_i \end{bmatrix},$$

$$\theta_i \neq k\pi/4, k \in \mathbf{Z}. \quad (3)$$

The interpretation is as follows: Let A be the polyphase matrix of a PR biorthogonal filter bank that fulfills the conditions for filter banks of type 1. Then multiplication by a factor S_i gives any other PR filter bank that fulfills conditions

for type 1 and has filters of the same length as A . Multiplication by a factor $\Lambda(z)$ increases the length of both filters by 2 while at the same time retaining the conditions for type 1.

Thus if we choose A to be a polyphase matrix of the shortest possible type 1 filter bank with difference l between the lengths of $H(z)$ and $G(z)$, we can generate all type 1 filter banks with filters of the same length difference l as in the initial filter bank. To assure that $H_p(z)$ describes a PR filter bank, every factor must not be singular, thus condition (3).

In [5, 6] only one possible matrix A is given:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (4)$$

This choice leads to filter banks which can be efficiently implemented by a lattice structure. However due to the singularity constraint (3) some classes of filter banks cannot be generated when only allowing (4) as a choice for A . Especially those where low- and high-pass filters differ in length are not covered by that choice for A . In order to significantly reduce the number of non-generatable filter banks we are interested in a larger class of possible choices here. Furthermore it would be desirable to have a condition available for generating only filter pairs of a certain length difference.

Our idea is to find an algorithm which can produce initial polyphase matrices A with arbitrary length differences $4K$ between the high- and low-pass filters. For this purpose we distinguish two cases:

- A. The symmetric filter is shorter than the antisymmetric filter

The shortest symmetric filter is $H(z) = 1 + z^{-1}$ (except for scaling). A general polyphase matrix with linear phase filters and length differences $4K$ in this case is of the form

$$A = \begin{bmatrix} 1 & 1 \\ \sum_{i=0}^{2K} a_i z^{-i} & \sum_{i=0}^{2K} -a_{2K-i} z^{-i} \end{bmatrix}, \quad a_0 \neq 0, a_{2K} \neq 0.$$

In order to guarantee FIR filters on both the analysis and the synthesis side $\det A$ has to be a monomial. Calculation of $\det A$ gives

$$\begin{aligned} \det A &= \sum_{i=0}^{2K} -a_{2K-i} z^{-i} - \sum_{i=0}^{2K} a_i z^{-i} \\ &= - \sum_{i=0}^{2K} (a_i + a_{2K-i}) z^{-i}. \end{aligned}$$

Thus $\det A$ is a polynomial with symmetric coefficients. To be a monomial only the middle coefficient can be unequal to zero. Thus for $K > 0$ we get the condition

$$\begin{aligned} \det A &= cz^{-r}, \quad c \neq 0 \\ \iff a_i &= -a_{2K-i}, \quad i \in \{0, \dots, K-1\} \quad \wedge \quad a_K \neq 0 \end{aligned}$$

and for $K = 0$ the condition is $a_0 \neq 0$.

This leads to the following structure for A:

$$A = \begin{bmatrix} 1 & 1 \\ P_{0K}(z) & P_{1K}(z) \end{bmatrix}, \quad (5)$$

$$P_{0K}(z) = \begin{cases} 1 & \text{if } K = 0 \\ 1 + \tan \beta_1 z^{-1} - z^{-2} & \text{if } K = 1 \\ P_{01}(z^K) + \sum_{j=2}^K z^{j-K-1} q_j(z) & \text{if } K \geq 2, \end{cases}$$

$$P_{1K}(z) = \begin{cases} -1 & \text{if } K = 0 \\ 1 - \tan \beta_1 z^{-1} - z^{-2} & \text{if } K = 1 \\ P_{11}(z^K) + \sum_{j=2}^K z^{j-K-1} q_j(z) & \text{if } K \geq 2 \end{cases}$$

and

$$q_j(z) = \tan \beta_j - \tan \beta_j z^{2(1-j)}.$$

The number of parameters β_j is equivalent to the factor K , $\beta_j \neq (2k+1)\pi/2, k \in \mathbf{Z}$ and in addition β_1 should be $\neq k\pi, k \in \mathbf{Z}$ to prevent A being singular.

B. The antisymmetric filter is shorter than the symmetric filter

The structure is very similar to the previous one. The shortest antisymmetric filter is $G(z) = 1 - z^{-1}$. The same strategy as in the previous case leads here to the structure

$$A = \begin{bmatrix} Q_{0K}(z) & Q_{1K}(z) \\ 1 & -1 \end{bmatrix}, \quad (6)$$

$$Q_{0K}(z) = P_{0K}(z) \text{ and } Q_{1K}(z) = -P_{1K}(z).$$

for length differences $4K$. Again the number of parameters β_j is equivalent to the factor K , $\beta_j \neq (2k+1)\pi/2, k \in \mathbf{Z}$ and β_1 should be $\neq k\pi, k \in \mathbf{Z}$ to prevent A being singular.

Note that in both cases, A and B, all Matrices A from (5) and (6) with $K \geq 1$ cannot be generated in any way by the original lattice product form.

To prove that these choices of A indeed lead to type 1 filter banks, we have to examine the lattice product $S = S_L \Lambda(z) S_{L-1} \Lambda(z) \cdots S_1 \Lambda(z) S_0$. One can show that this structure gives a matrix

$$S = \begin{bmatrix} a(z) & z^{-L} b(z^{-1}) \\ b(z) & z^{-L} a(z^{-1}) \end{bmatrix}. \quad (7)$$

An arbitrary polyphase matrix of a type 1 filter bank with length difference $4K$ between low- and high-pass filters is of the form

$$H_p(z) = \begin{bmatrix} x(z) & z^{-N} x(z^{-1}) \\ y(z) & -z^{-N-2K} y(z^{-1}) \end{bmatrix} \quad (8)$$

for case A, where N is half the degree of the shorter filter of the bank. When multiplying (5) and (7) it is easy to show that this indeed leads to a matrix of the form (8). Case B is analogous.

Now we have a simple structure to systematically generate a larger class of type 1 PR filter banks than it is possible by the original structure with (4) as the only choice for A . Some still cannot be generated by this structure due to the singularity constraint (3) (see [6] for an example), but these cases seem to be isolated and can be approximated by the structure because of the continuity of $\tan \beta_j$.

As mentioned above we have to assure that $H(z)$ is a low-pass and $G(z)$ a high-pass to be certain that the two filters represent a discrete scaling function and mother wavelet respectively. Fortunately it can be shown that these conditions are fulfilled for any choice of parameters.

3.2. The Odd-length Case

In [5] also an algorithm for calculating a lattice factorization of a given polyphase matrix with filters of type 2 is presented. This algorithm cannot be used directly to systematically generate the desired polyphase matrices, but we can use the results to develop an algorithm which is similar to the one in the even-length case.

That means, that we are interested in a product structure where any additional factor increases the lengths of both filters by two while preserving the conditions for type 2 at the same time. As in the even-length case we then only have to find an algorithm for generating an initial polyphase matrix which can afterwards be extended arbitrarily by use of the product form.

One can easily derive this product form from [5, 6]:

$$H_p(z) = A \prod_{i=1}^L \begin{bmatrix} 1 + z^{-1} & 1 \\ 1 + z^{-1} \tan \theta_i + z^{-2} & 1 + z^{-1} \end{bmatrix} \cdot \begin{bmatrix} \sin \delta_i & 0 \\ 0 & \cos \delta_i \end{bmatrix}. \quad (9)$$

If A corresponds to length 3 and length $5 + 4K$ symmetric PR filters then $H_p(z)$ corresponds to length $3 + 2L$ and length $5 + 4K + 2L$ PR filters, provided that $\sin \delta_i \neq 0$ and $\cos \delta_i \neq 0$. In some cases however the left- and rightmost coefficients could become zero. To avoid this we can calculate the condition $\tan \delta_i \neq -1$. To assure furtheron that $H_p(z)$ is invertible additionally $\tan \theta_i$ has to be $\neq 2$.

A shortest symmetric filter of odd length is $1 + \tan \alpha z^{-1} + z^{-2}$. Using a filter of length $N = 3 + 4K + 2$ with arbitrary coefficients as the second filter we can formu-

late the general polyphase matrix for this case

$$A = \begin{bmatrix} 1 + z^{-1} & \tan \alpha \\ \sum_{i=0}^{K+1} b_i z^{-i} & \sum_{i=0}^K c_i z^{-i} \\ + b_i z^{i-2(K+1)} & + c_i z^{i-2K-1} \end{bmatrix}. \quad (10)$$

Calculating the determinant of (10) and imposing that it should be a monomial finally leads to the structure

$$A = \begin{bmatrix} 1 + z^{-1} & \tan \alpha \\ P_{0K}(z) & P_{1K}(z) \end{bmatrix}, \quad (11)$$

$$P_{0K}(z) = \sum_{i=0}^{K+1} \tan \beta_i z^{-i} + \tan \beta_i z^{i-2K-2},$$

$$P_{1K}(z) = \sum_{i=0}^K a_i z^{-i} + a_i z^{i-2K-1},$$

$$a_i = \tan \alpha \sum_{j=0}^i (-1)^{i+j} \tan \beta_j \quad i < K \quad \text{and}$$

$$a_K - \tan \alpha \tan \beta_{K+1} \neq 0.$$

So we need $K+4$ parameters (α, a_K and $\beta_i, i = 0, \dots, K+1$) here and the β_i have to be $\neq (2k+1)\pi/2, k \in \mathbf{Z}$.

To prove that (9) and (11) generate type 2 PR filter banks assume an arbitrary polyphase matrix $H_p(z)$ of such a filter bank and multiply it by the inverse of (11). We then receive a polyphase matrix where both filters are still symmetric and whose lengths differ by 2. It is proven in [5] that every polyphase matrix of this kind can be generated by the product structure (9) (except for scaling). Thus in the odd-length case our product structure is a complete characterization of type 2 filter banks.

Unlike the even-length case it is not structurally guaranteed by (11) that one row always represents a low-pass and the other row a high-pass. Thus the question arises if it is possible to fulfill the low-pass/high-pass condition by imposing constraints on the parameters.

There are two possible solutions to this problem. If we are only interested in filters of a certain length we can calculate the values of the parameters θ_L, δ_L of the last factor in the lattice product depending on the filters resulting from the lattice product without the last factor.

If we are interested in receiving only low-/high-pass pairs at every stage of the lattice product, every pair of parameters has to fulfill one of the following conditions

- (a) $3 \cot \delta_i - 4 = \tan \theta_i,$
- (b) $1 - \cot \delta_i = \tan \theta_i,$
- (c) $-3 \cot \delta_i - 4 = \tan \theta_i,$
- (d) $\cot \delta_i = \tan \theta_i$

and the parameters of A have also to be chosen so that the corresponding filters constitute low- and high-pass filters.

4. CONCLUSION

Our aim has been to develop an algorithm for systematically generating as many finite discrete wavelets with linear phase as possible. For this purpose we adapted the well-known lattice factorizations for PR linear phase FIR filter banks. Because we neglected the implementation point of view we were able to eliminate the disadvantage of the inability to describe filter pairs with length differences in the even-length case. Thus we could enlarge the class of parameterized filter banks in that case significantly and could also focus on the systematic generation instead of implementation of the desired filter banks in both cases. As a result we got algorithms for systematically generating filter pairs of given lengths and length differences.

In the even-length case all generated filters automatically fulfill the conditions for discrete scaling functions and wavelets and in the odd-length case we were able to give the necessary constraints on the parameters to achieve this goal.

5. REFERENCES

- [1] H. Zou and A. H. Tewfik, "Parametrization of Compactly Supported Orthonormal Wavelets," *IEEE Transaction on Signal Processing*, vol. 41, no. 3, pp. 1428–1431, 1993.
- [2] D. Pollen, " $SU_1(2, F[z, 1/z])$ for F a Subfield of $\mathbf{C}$," *Journal of the AMS*, vol. 3, no. 3, pp. 611–624, 1990.
- [3] O. Rioul, "A Discrete-Time Multiresolution Theory," *IEEE Transactions on Signal Processing*, vol. 41, pp. 2591 – 2606, August 1993.
- [4] A. Cohen, I. Daubechies, and J.-C. Feauveau, "Biorthogonal Bases of Compactly Supported Wavelets," *Communications on Pure and Applied Mathematics*, vol. XLV, pp. 485–560, 1992.
- [5] T. Q. Nguyen and P. P. Vaidyanathan, "Two-Channel Perfect-Reconstruction FIR QMF Structures Which Yield Linear-Phase Analysis and Synthesis Filters," *IEEE Transactions on Acoustics, Speech and Signal Processing*, vol. 37, pp. 676–690, May 1989.
- [6] M. Vetterli and D. L. Gall, "Perfect Reconstruction FIR Filter Banks: Some Properties and Factorizations," *IEEE Transactions on Acoustics, Speech and Signal Processing*, vol. 37, pp. 1057–1071, July 1989.