CMA CONVERGENCE FOR CONSTANT ENVELOPE, NON-ZERO BANDWIDTH SIGNALS

Sally L. Wood[1] *Michael G. Larimore*[2] *John R. Treichler*[2]

[1] Electrical Engineering Department, Santa Clara University
Santa Clara, California 95053 USA
[2] Applied Signal Technology, Sunnyvale, California USA

ABSTRACT

The gradient-descent-based Constant Modulus Algorithm (CMA) is commonly thought to converge much more slowly than its Least Mean Square (LMS) counterpart, particularly for quadrature-amplitude-modulated (QAM) signals. Experiments shown in this paper indicate that in fact there is no substantial difference in the convergence rates of the two methods in the important special case of constant envelope signals (e.g., FM and FSK). For both CMA and LMS algorithms the convergence for nontrivially frequency-modulated signals depends on the same eigenvalue disparity problem that affects all gradient-descent techniques.

1. INTRODUCTION

This paper addresses the common perception that the Constant Modulus Algorithm (CMA) converges slowly. One implication of this perception is a common belief that CMA should only be used as a last resort or an interim step in the design of digital and analog communications systems. Another implication is that new, better, and, in particular, faster blind adaptive algorithms need to be developed to replace CMA. In this paper we show that in fact CMA converges at approximately the same asymptotic rate as an LMS-adapted filter for constant envelope inputs.

2. BACKGROUND

The beginning of modern data communications can be marked by Bob Lucky's 1965 BSTJ paper on adaptive equalization. The concept of using an adaptive digital filter to compensate for the distortion introduced by a telephone channel led to significant increases in the transmission rates achievable over that medium. We should note that Lucky's equalizer was "blind", and did not need to know the transmitted data sequence in order to adapt the filter's coefficients. The simple nature of the modulation in the telephone modems of the day made this possible, but his "zero-forcing" algorithm proved inadequate when the modulation complexity was increased to QPSK and 8-PSK to improve data throughput.

To deal with these higher-order modulations, the concept of equalizer training was introduced. A known sequence was transmitted at the begining of a communications session to "train" the equalizer. Since most digital communications systems were of a point-to-point nature, the use of training procedures upon link initialization was at the time both accepted and effective. Blind techniques were first suggested as solutions to equalization problems in high-rate digital communications in the early 1980s [1] [2]. For ten years or so, interest in these techniques languished. Part of the reason for this was that virtually all digital communications systems at the time were still point-to-point. In fact a large measure of the current interest in blind equalization stems from the desire to use digital modulation in point-to-multipoint and broadcast applications in which "training at startup" is impractical or impossible. An important practical example is the use of digital transmission for advanced and "high-definition" television distribution.

The second reason for adhering to the tradition of using training is the perception that blind equalization techniques converge orders of magnitude more slowly than RLS- and even LMS-based filters. To remedy this perceived problem, researchers have suggested a number of alternatives, including, among others, the use of block and RLS-like adaptation (e.g., [3]), filtered-regressor and quasi-Newton approaches [7], and abandonment of the CMA cost function in favor of higher- and second-order-statistical methods [6]. The study presented here takes another course, that of showing what factors limit the convergence performance of a CMA-based equalizer. In this paper we show that in certain important cases CMA converges at exactly the same asymptotic rate as LMS. This approach also lays the groundwork for understanding the converence speed limitations in the case of QAM signals.

3. TECHNICAL APPROACH

The convergence behavior of a CMA-directed filter can be analyzed in three distinct steps: its response to sinusoidal inputs, its response to constant envelope signals with non-zero bandwidth (such as FM, FSK, and n-PSK signals), and its response to non-constant envelope, non-zero bandwidth signals, such as QAM. The first step was analyzed in [5], using an adaptation model developed in [4] to explain CMA's signal capture behavior. In light of the perceived "slowness" of CMA, [5] reported the following surprising results: (1) the eigenfilter corresponding to the sinusoidal input captured by a CMA-directed filter converged *at exactly the same rate* as it would have in an LMS-directed filter, and (2) the eigenfilter associated with a sinusoid being suppressed by the CMA-based actually converged (to zero) *at twice the rate* induced by an LMS-directed filter. Thus, far from converging slower than LMS, the CMA-directed filter reacted to isolated, statistically independent narrowband inputs as fast or faster than LMS did.

This paper addresses the second step, the extension of the work in [5] to the case of non-zero bandwidth, constant-envelope signals. Specifically we seek to show that:

- In response to an FM signal, a CMA-based filter of a given length and with a given adaptation constant converges *at the same asymptotic rate* as an LMS-based filter,
- The asymptotic convergence rate of the CMA-adapted filter is limited by the eigenvalue disparity of the input signal in exactly the same way and by the same amount as an LMS-based filter, thus strongly suggesting the utility of RLS-style recursions for this case.

Modulating signals include bandlimited noise, smaller sums of sinusoids with frequency differences below the frequency resolution limit of the filter, and sums of sinusoids with frequencies on the bin numbers for the selected filter lengths. The last case is the easiest to analyze in terms of initial convergence rates, but it is also the most likely to converge to a suboptimal filter.

For both the CMA and LMS cases the adaptive filter output $y(k)$ is given by

$$y(k) \ = \ \underline{w}_k^H \cdot \underline{x}_k \tag{1}$$

where the superscript H indicates Hermitian transpose. The complex LMS algorithm uses $e(k) = d(k) - y(k)$ for the error function with a cost function equal to half the expected value of the squared magnitude of the error. The resulting update equation is given by

$$\underline{w}_{k+1} \ = \ \underline{w}_k \ + \ \mu e^*(k)\underline{x}_k \tag{2}$$

The error function $e(k) = 1 - ||y(k)||^2$ for the CMA algorithm is used with a cost function equal to one fourth of the expected value of the squared magnitude of the error. The filter coefficients are updated using

$$\underline{w}_{k+1} \ = \ \underline{w}_k \ + \ \mu e(k) y^*(k)\underline{x}_k \tag{3}$$

4. CMA VS LMS FOR A RANDOMLY MODULATED FM SIGNAL

Frequency-modulated signals were used as the filter inputs for comparison of the convergence performance of LMS- and CMA-directed filters under computer simulation. The input to the modulator was constructed to have a bandlimited noise-like nature by combining a large number of equal-amplitude sinusoids with randomized initial phases. This nearly Gaussian, bandlimited signal was scaled by a selected modulation index and then applied to a frequency modulator. An example of such a signal is shown in Figure 1. A segment of the real part of the signal for a duration equal to a filter length is shown in Figure 1a. The corresponding spectrum is shown in Figure 1b. For both the LMS and CMA cases the modulated signal was applied to an N-tap FIR adaptive filter. For all cases reported here $N = 128$, and a normalized sampling interval of 1 was used. The normailzed carrier frequency was .0625.

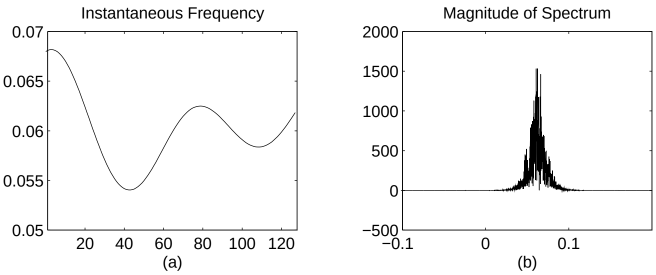

Figure 1. A Typical Modulating Signal Segment (real part) and the Signal Spectrum

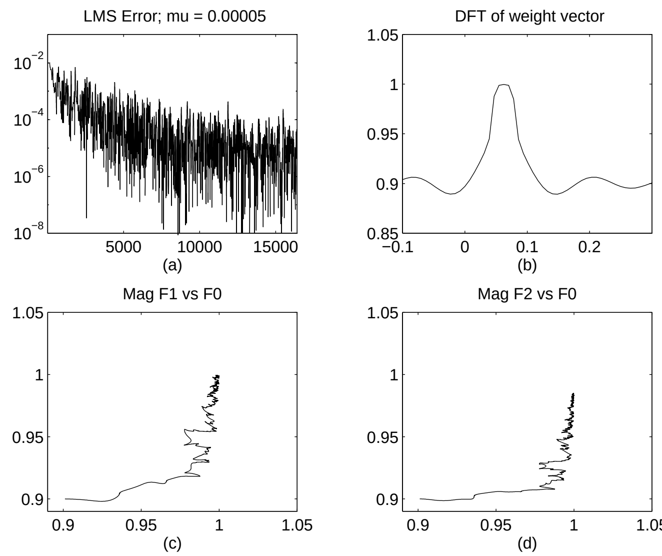

Figure 2. Typical Convergence Behavior for an LMS-directed Filter with a Noise-loaded FM Input

For the LMS case, the desired signal $d(k)$ was obtained by delaying the modulator output by 8 samples. For CMA, of course, no reference signal was needed. The filter initial condition was chosen in all cases to be a unit pulse at the eighth filter tap (thus corresponding the delay chosen for the LMS filter) but the amplitude of the pulse was chosen in various ways during the experiments. The rationale for choosing an initial condition close to the desired objective for both LMS and CMA was a desire to focus on the performance near final convergence. Figures 2 and 3 show a typical set of comparative results. In both cases the filter was initialized with a pulse amplitude of 0.9 and the input signals were as shown in Figure 1.

Figure 2 shows the response of the LMS directed filter. The "learning curve" is shown in Figure 2a, and the magnitude of the filter transfer function after over 16000 iterations is shown in Figure 2b. In addition, values of the magnitude of the transfer function are plotted against each other over time. In Figure 2c the magnitude of the transfer function at $f_c + \frac{1}{N}$ is plotted against the magnitude at f_c. The same type of plot in Figure 2d shows the magnitude at $f_c + \frac{2}{N}$ plotted against the magnitude at f_c.

Figure 3 shows the same type of data for an CMA-

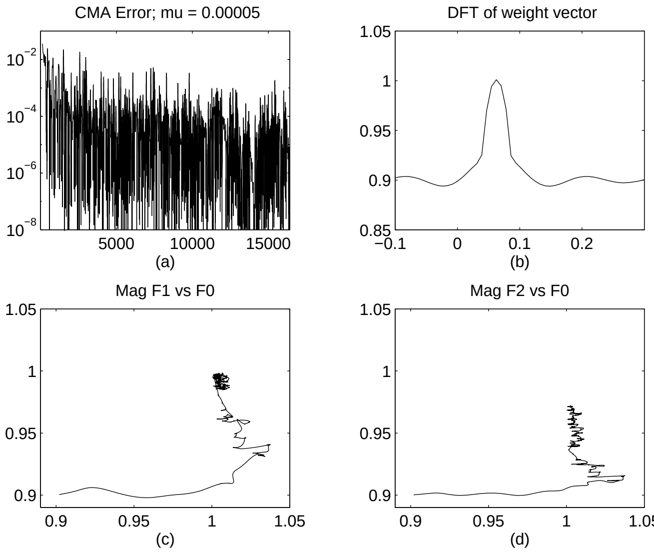

Figure 3. Typical Convergence Behavior for a CMA-directed Filter with a Noise-loaded FM Input

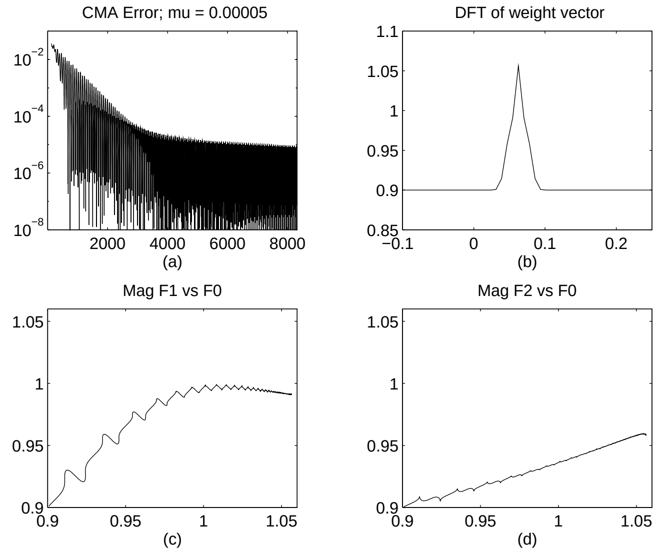

Figure 4. Convergence of CMA for Sinusoidal Modulating Signal with Frequency 1/N.

directed filter with an FM input constructed in the same way. The filter transfer functions are similar. However, it is clear from a comparison of the two lower plots of each figure that the trajectories for the different frequency components of the two filters is not the same. The value at f_c for the CMA filter overshoots and then conveges back toward the ideal value while in the LMS filter the value approaches the ideal value from below.

5. SOME SPECIAL CASES FOR PERIODIC INPUTS

When first analyzing the convergence of CMA, it is tempting to use a simple sinusoidally modulated FM signal. This type of signal is particularly attractive if the frequency of the modulating sinusoid is chosen to be an integer multiple of $\frac{f_s}{N}$, where f_s is the sampling frequency and N is the length of the adaptive filter. When chosen this way the Bessel structure of the FM signal's spectrum has the property that all of the signal's energy is also concentrated at integer multiples of $\frac{f_s}{N}$ and the filter gains at all of those frequencies are easily ascertained with an N-point DFT of the adaptive filter's tap weights. In this case is also tempting to try to apply the results obtained in [4] to predict the convergence rates of the various tonal components of the FM signal.

Figure 4 shows the results obtained from CMA using a sinysoidal modulating functionwith $\beta = 1.6$. The magnitudes of the lines in the spectrum of the modulated signal starting from f_c are given by the Bessel function values for this β as .4554, .5699, .2570, and .0725. Using the power ratios of these lines, the initial slopes in Figures 4c and 4d are very close to the respective predicted values of 1.57 and .32. As in the case shown in Figure 3, the component at f_c overshoots its ideal value. The behavior shown in Figure 4 is consistent over a wide range of values for the adaptation constant μ and over a wide range of initial filter values less than 1. For this case of a sinusoidal modulating function, the LMS output starts with the same trajectories as the CMA filter. However the overshoot at f_c does not occur, and the resulting filter is flatter and wider. After 8000 iterations, the CMA filter produced a squared error lower than the LMS filter by a factor of 10. When the CMA filter is initialized to a value greater than 1, the component at f_c is attenuated instead of amplified.

When the modulating signal was the sum of two sinusoids at frequencies of $\frac{1}{N}$ and $\frac{1}{3N}$, the behavior of the LMS and CMA filters was similar and predictable. However, when the two frequecies were $\frac{1}{N}$ and $\frac{3}{N}$ so that the modulating waveform had a period equal to the filter length, the unexpected results shown in Figure 5 were obtained.

The source of both of these problems is the same. When the input to the modulator is chosen to have a period exactly equal to a submultiple of the filter length, then both it and the FM signal are periodic in the filter length, and the resulting data matrix is circulant. It is easily shown that the weight vector $\underline{w}$ can be selected to produce any of a variety of almost constant-envelope signals. The one chosen by the adaptive filter will depend strongly on the filter's initial condition. Thus different initial conditions will produce different convergent filters even for the same input signal.

Now why does the CMA filter not capture as we expected? In [4] it was assumed that the input sinusoids are statistically independent in that they have random starting phases. In the case of the FM signal the phases of the various Bessel-weighted sinusoids are deterministically related. It should also be noted that the experimental results cited in [8] showed that QAM inputs with period equal to the equalizer length also caused unusual and undesirable convergence behavior.

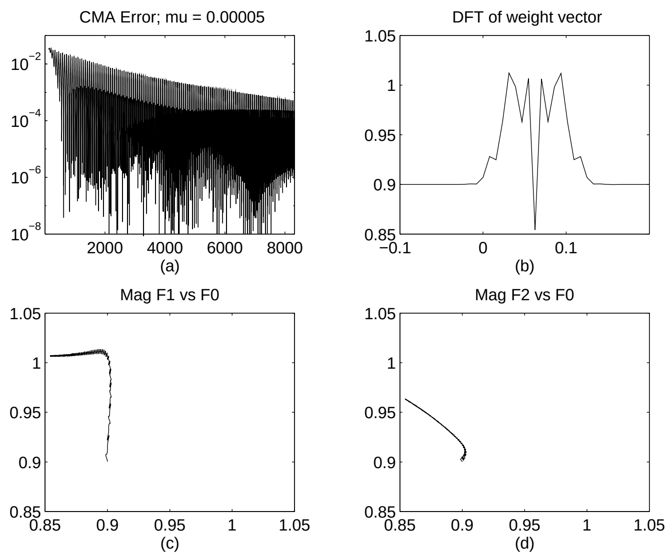

Figure 5. Misconvergence of CMA for Modulating Signal Composed of two Sinusoidal Signals at Frequencies of 1/N and 3/N.

6. SUMMARY

The results presented here show that the convergence rates of CMA and LMS algorithms are comparable for nonzero bandwidth constant modulus signals which are not periodic with a period equal to a submultiple of the filter length. Attention should now be focused on pulsed signals with discrete inputs of the type used to transmit digital data, e.g. PSK and QAM. This may be done in two steps.

- Analyze the CMA-directed asymptotic convergence rate of a QPSK signal in the important special case of a $\frac{T}{2}$-spaced equalizer, comparing it again to that of a training-directed LMS equalizer. Since QPSK is a constant envelope signal when sampled at "top dead center", the asymptotic dynamics of a CMA-directed equalizer can be expected to be controlled by the eignevalues of the input correlation matrix. This implies that the CMA-directed equalizer will converge at the same rate as an LMS-directed equalizer and its misadjustment can be expected to be about the same (for the same filter length and adaptation constant μ).
- Extend the analysis for QPSK to the last step, the case of non-constant envelope, non-zero bandwidth signals.

Recent as-yet-unpublished work by Fijalkow, et al [9] shows that the misadjustment for a CMA-directed filter is the same as that of an LMS-directed filter plus a term strongly proportional to the degree that the input signal is non-constant envelope. (Thus a PSK signal would expect to see the same misadjustment with either an LMS or CMA-directed filter since it has a constant envelope.) Since misadjustment noise contributes to the overall noise at the output of the adaptive filter, this suggests that the adaptation constant μ for the CMA-directed filter might be typically set much lower than that of an LMS-directed filter in order to attain the same output SNR or MSE. Setting the adaptation constant substantially smaller would, of course, slow down convergence by the same factor. Is this misadjustment-driven reduction in μ in fact the reason that CMA is considered to be slower? The careful analytical work needed to address the QPSK and QAM convergence rate questions listed above will answer this question as well.

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