HARD-CONSTRAINED SIGNAL FEASIBILITY PROBLEMS*

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ABSTRACT

We consider the problem of synthesizing feasible signals in the presence of inconsistent convex constraints, some of which are hard in the sense that they must absolutely be satisfied. This problem is formalized as that of minimizing an objective function measuring the degree of unfeasibility with respect to the soft constraints over the intersection of the sets associated with the hard constraints. We first investigate the process of aggregating soft constraints in order to define relevant objectives and then address the question of solving the resulting convex programs. Finally, we provide numerical results to illustrate the benefits of our analysis.

INTRODUCTION

The goal of a set theoretic signal synthesis (estimation or design) problem is to produce a signal $a^\star$ consistent with a family $(\Psi_i)_{i\in I}$ of constraints. Each constraint Ψ_i is associated with a set $S_i=\{a\in\Xi \mid a \text{ satisfies } \Psi_i\}$ in a suitable signal space Ξ. The feasibility problem is then stated as

$$\text{Find } a^\star \in S \triangleq \bigcap_{i\in I} S_i. \tag{1}$$

Throughout this paper, Ξ is a real Hilbert space with distance d, $I=\{1,\ldots,m\}$ is finite, and the S_i's are closed and convex. This convex set theoretic feasibility framework has been applied to a wide range of signal processing problems, e.g., [2]-[5], [9].

In certain problems (1) may not have solutions because incompatible constraints are present and, therefore, $S=\varnothing$ [2], [5]. In such instances, it was shown in [6] that the solutions produced by the popular POCS algorithm

$$(\forall n\in\mathbb{N})\ \ a_{n+1}=P_{n\ (\text{modulo } m)+1}(a_n), \tag{2}$$

where P_i is the projector onto S_i, are guaranteed to lie at best in one of the sets. They are therefore not reliable for it is not known whether they satisfy – in any approximate sense – the remaining $m-1$ constraints. A notable exception is when $m=2$, as the solutions can then be interpreted as points that satisfy one constraint and are closest to the set representing the other one. A more satisfactory approach was proposed in [2], where the parallel projections scheme

$$(\forall n\in\mathbb{N})\ \ a_{n+1}=a_n+\lambda_n\Big(\sum_{i\in I} w_i P_i(a_n)-a_n\Big), \tag{3}$$

with $(\lambda_n)_{n\geq 0}\subset[\varepsilon,2-\varepsilon]$ $(0<\varepsilon<1)$, $(w_i)_{i\in I}\subset\mathbb{R}_+^*$, and $\sum_{i\in I} w_i=1$, was shown to converge weakly to a minimizer of the proximity function $\Phi: a\mapsto\sum_{i\in I} w_i d(a,S_i)^2$, i.e., to a weighted least-squares solution of the inconsistent feasibility problem (1).

In this paper, we consider inconsistent signal feasibility problems in which some constraints – called "hard" hereafter – must absolutely be enforced, e.g., because they arise from certain *a priori* information in estimation problems or because they correspond to imperative specifications in design problems. Accordingly, the family $(\Psi_i)_{i\in I}$ is broken up into a group of hard constraints $(\Psi_i)_{i\in I^\blacktriangle}$ and a disjoint group of soft constraints $(\Psi_i)_{i\in I^\vartriangle}$. Now, let $S^\blacktriangle=\bigcap_{i\in I^\blacktriangle} S_i$ and let $\Phi^\vartriangle$ be some objective function aggregating the soft constraints $(\Psi_i)_{i\in I^\vartriangle}$. Then the hard-constrained version of the inconsistent signal synthesis problem (1) reads

$$\text{Find } a^\star\in G\triangleq\{a\in S^\blacktriangle \mid (\forall b\in S^\blacktriangle)\ \Phi^\vartriangle(a)\leq\Phi^\vartriangle(b)\}. \tag{4}$$

In the following sections, we first discuss the construction of $\Phi^\vartriangle$ and then the problem of solving (4). Finally, we show some numerical results.

SOFT CONSTRAINTS MODELING AND AGGREGATION

Throughout, we shall make the following assumptions.

1. $(\forall i\in I)\ \ S_i=\{a\in\Xi \mid g_i(a)\leq 0\}$, where the functions $(g_i)_{i\in I}$ are convex and continuous.
2. $S^\blacktriangle$ is (closed, convex) nonempty and bounded.
3. $\Phi^\vartriangle$ is convex and continuous.

It follows that G in (4) is nonempty, bounded, closed, and convex, and that $\Phi^\vartriangle$ and $(g_i)_{i\in I}$ are subdifferentiable [1].

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Besides the above mathematical requirements, the selection of a pertinent objective $\Phi^{\vartriangle}$ to aggregate the soft functions $(g_i)_{i\in I^{\vartriangle}}$ should also be guided by some commonsense conditions. Let $I^{\blacktriangle} = \{1,\ldots,k\}$ and $I^{\vartriangle} = \{k+1,\ldots,m\}$. Then, for some function $\varphi : \mathbb{R}^{m-k} \to \mathbb{R}$, $\Phi^{\vartriangle} \triangleq \varphi(g_{k+1},\ldots,g_m)$ and we shall require that the conditions below be met.

1. φ is invariant under permutations of its $m-k$ variables.
2. $(\forall a \in \Xi)\ \Phi^{\vartriangle}(a) \geq 0$ and $\Phi^{\vartriangle}(a) = 0 \Leftrightarrow a \in S^{\vartriangle} \triangleq \bigcap_{i\in I^{\vartriangle}} S_i$. Consequently, φ is nonnegative and vanishes only on the nonpositive orthant $(\mathbb{R}_-)^{m-k}$.
3. Monotonicity: when $m-k-1$ of its variables are held fixed, φ is an increasing [resp. constant] function of the remaining variable on $\mathbb{R}_+$ [resp. on $\mathbb{R}_-$].
4. The subgradients of $\Phi^{\vartriangle}$ are easy to compute.

Now let $(\forall(i,a) \in I^{\vartriangle}\times\Xi)\ g_i^+(a) \triangleq \max\{0, g_i(a)\}$ and let $(f_i)_{i\in I^{\vartriangle}}$ be increasing convex functions from $\mathbb{R}_+$ into $\mathbb{R}_+$ that vanish at 0. Then two generic aggregating functions are

$$\begin{cases} \Phi_1^{\vartriangle} : a & \mapsto & \sum_{i\in I^{\vartriangle}} f_i \circ g_i^+(a) \\ \Phi_2^{\vartriangle} : a & \mapsto & \max_{i\in I^{\vartriangle}} f_i \circ g_i^+(a). \end{cases} \tag{5}$$

Let us note in passing that the proximity function $\Phi^{\vartriangle} : a \mapsto \sum_{i\in I} w_i d(a,S_i)^2$ $((w_i)_{i\in I} \subset \mathbb{R}_+^*)$ – used in [2] when $I^{\blacktriangle} = \varnothing$ – appears as a special case of $\Phi_1^{\vartriangle}$.

NUMERICAL ALGORITHMS

We now address the problem of solving the hard-constrained signal feasibility problem (4). Let us note that the algorithms mentioned in the Introduction solve (4) only in two cases. POCS (2) applies when $m = 2$, $I^{\blacktriangle} = \{1\}$, $I^{\vartriangle} = \{2\}$, and $\Phi^{\vartriangle}(a) = d(a,S_2)$. On the other hand, (3) applies when $I^{\blacktriangle} = \varnothing$ and $\Phi^{\vartriangle}(a) = \sum_{i\in I} w_i d(a,S_i)^2$. While there is no best approach to solve (4), several methods are available which are suitable in specific contexts. We present here a few examples. Hereafter, $P^{\blacktriangle}$ denotes the projector onto $S^{\blacktriangle}$, $\alpha^\star = \inf_{a\in S^{\blacktriangle}} \Phi^{\vartriangle}(a)$, and it is assumed that $\arg\min_{a\in\Xi} \Phi^{\vartriangle}(a) \notin S^{\blacktriangle}$.

Algorithm 1 [8] Suppose that $\Phi^{\vartriangle}$ is differentiable on $S^{\blacktriangle}$, that $\nabla\Phi^{\vartriangle}$ has Lipschitz constant M on $S^{\blacktriangle}$, and that $0 < \varepsilon < 1$. Given $a_0 \in S^{\blacktriangle}$, define

$$(\forall n \in \mathbb{N})\ \ a_{n+1} = P^{\blacktriangle}(a_n - \lambda_n \nabla\Phi^{\vartriangle}(a_n)), \tag{6}$$

where $(\lambda_n)_{n\geq 0} \subset [\varepsilon, 2/(M+2\varepsilon)]$. Then $(\Phi^{\vartriangle}(a_n))_{n\geq 0}$ converges to $\alpha^\star$; if Q is strictly [resp. uniformly] convex, then (4) admits a unique solution $a^\star$ and $(a_n)_{n\geq 0}$ converges weakly [resp. strongly] to $a^\star$. □

The above algorithm is applicable when $P^{\blacktriangle}$ is easily computable, i.e., when $S^{\blacktriangle}$ is geometrically simple. It moreover requires that $\Phi^{\vartriangle}$ be differentiable on $S^{\blacktriangle}$. In the next algorithm these restrictions are lifted but it is assumed that a reasonably tight upper bound $\overline{\alpha}$ is available for $\alpha^\star$. The corresponding approximation to problem (4) is then

$$\text{Find } a^\star \in \widetilde{G} \triangleq \{a \in \Xi \mid \Phi^{\vartriangle}(a) \leq \overline{\alpha}\} \cap S^{\blacktriangle}, \tag{7}$$

which is a consistent 2-set feasibility problem. Now construct a function $\Phi^{\blacktriangle}$ aggregating the hard functions $(g_i)_{i\in I^{\blacktriangle}}$ according to the same conditions as for $\Phi^{\vartriangle}$ above. Then

$$\widetilde{G} = \{a \in \Xi \mid \Phi^{\vartriangle}(a) \leq \overline{\alpha}\} \cap \{a \in \Xi \mid \Phi^{\blacktriangle}(a) \leq 0\} \tag{8}$$

and (7) can be solved via an extrapolated subgradient projections method.

Algorithm 2 [4] With the convention $0/0 = 1$ in force, let $P_n^{\blacktriangle}(a_n) = a_n - \Phi^{\blacktriangle}(a_n)t_n^{\blacktriangle}/\|t_n^{\blacktriangle}\|^2$, where $t_n^{\blacktriangle}$ is a subgradient of $\Phi^{\blacktriangle}$ at a_n (an analogous definition applies to $P_n^{\vartriangle}(a_n)$). Given $a_0 \in \Xi$, define for every $n \in \mathbb{N}$

$$a_{n+1} = a_n + \rho_n(P_n^{\blacktriangle}(a_n) + P_n^{\vartriangle}(a_n) - 2a_n), \tag{9}$$

where

$$\rho_n = \frac{\|P_n^{\blacktriangle}(a_n) - a_n\|^2 + \|P_n^{\vartriangle}(a_n) - a_n\|^2}{\|P_n^{\blacktriangle}(a_n) + P_n^{\vartriangle}(a_n) - 2a_n\|^2}. \tag{10}$$

Suppose that the subgradients of $\Phi^{\blacktriangle}$ and $\Phi^{\vartriangle}$ are uniformly bounded on bounded sets. Then $(a_n)_{n\geq 0}$ converges weakly to a point $a^\star \in \widetilde{G}$. □

It should be noted that Algorithm 2 solves (4) exactly when $\overline{\alpha} = \alpha^\star$ and gives an approximate solution otherwise. In some cases, it is possible to refine $\overline{\alpha}$ over the iterations. Results in that direction can be found in [7].

The third algorithm is a cutting-plane method.

Algorithm 3 Let Q_0 be a bounded polyhedron (finite intersection of closed half-spaces) containing $S^{\blacktriangle}$. Suppose that $\Phi^{\vartriangle}$ is uniformly convex on Q_0 and that the subgradients of the k hard functions $(g_i)_{i\in I^{\blacktriangle}}$ are uniformly bounded on Q_0 (e.g., Ξ has finite dimension and $\Phi^{\vartriangle}$ is strictly convex on Q_0), and let $a_0 = \arg\min_{a\in Q_0} \Phi^{\vartriangle}(a)$. A sequence $(a_n)_{n\geq 0}$ is constructed as follows. At iteration $n \in \mathbb{N}$, $i(n) = n$ (modulo k) + 1, $t_{i(n)}$ is a subgradient of $g_{i(n)}$ at a_n, $H_n = \{a \in \Xi \mid \langle a_n - a \mid t_{i(n)}\rangle \geq g_{i(n)}(a_n)\}$, and $Q_{n+1} = Q_n \cap H_n$. Now define the new iterate as $a_{n+1} = \arg\min_{a\in Q_{n+1}} \Phi^{\vartriangle}(a)$. Then $(a_n)_{n\geq 0}$ converges strongly to the unique solution $a^\star$ of (4). □

In Algorithm 3, only the ability to compute the subgradients of the hard functions and to solve linear programs is required. However, as n increases, the complexity of the polyhedron Q_n grows and so does that of the linear subprograms. Variants of Algorithm 3 can be devised that mitigate or eliminate altogether this problem. We shall present these results elsewhere.

SIMULATION RESULTS

We present here a simple application of the proposed framework in which we revisit the digital pulse shape design problem of [2]. In this problem four incompatible constraints are present:

Ψ_1: The Fourier transform of the pulse is zero at multiples of 50Hz and beyond 300Hz.

Ψ_2: The pulse has linear phase and its midpoint has amplitude 1.

Ψ_3: The energy of the pulse does not exceed $\xi = 4$.

Ψ_4: The actual duration of the pulse is 50 ms and it has periodic zero crossings every 3.91 ms.

The associated property sets $(S_i)_{1\leq i\leq 4}$ and their projectors can be found in [2]. In the first experiment, no hard constraint is imposed and a least-squares cost is chosen to aggregate the soft constraints, namely, $\Phi^\Delta : a \mapsto (1/4)\sum_{i=1}^4 d(a,S_i)^2$. The results are shown in Figs. 1-2. Ψ_1 is then chosen as a hard constraint and the soft constraints are aggregated with $\Phi^\Delta : a \mapsto (1/3)\sum_{i=2}^4 d(a,S_i)^2$. The results are shown in Figs. 3-4. Next, Ψ_4 is chosen as a hard constraint and the soft constraints are aggregated with $\Phi^\Delta : a \mapsto (1/3)\sum_{i=1}^3 d(a,S_i)^2$. The results are shown in Figs. 5-6. Since Φ^Δ has a lipschitzian gradient and all the projectors are easily computable, Algorithm 1 was used in the two hard-constrained problems.

At this point it is worth noting that a pulse satisfying Ψ_4 can also be obtained by implementing POCS as $(\forall n \in \mathbb{N})\ a_{n+1} = P_4 \circ P_1 \circ P_2 \circ P_3(a_n)$. However, as noted in [2], there is no guarantee that the solution thus obtained is close to the other sets in any sense. One can check in Figs. 7-8 that the pulse generated by POCS does indeed satisfy Ψ_4 but is worse than that produced in Figs. 5-6 in terms of satisfying the remaining constraints $(\Psi_i)_{1\leq i\leq 3}$.

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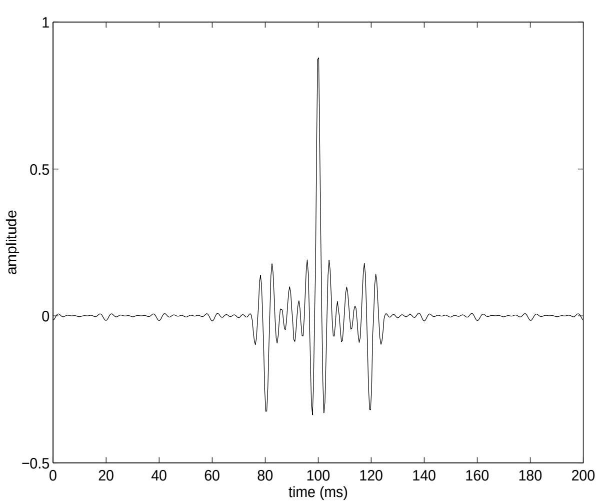

Fig. 1: Pulse generated without hard constraints.

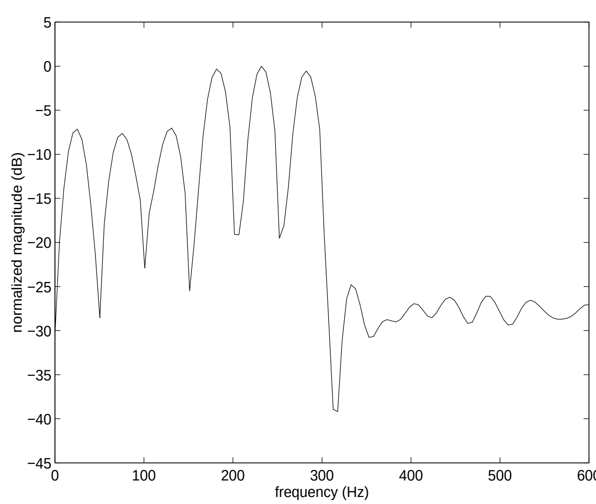

Fig. 2: Normalized spectral density of Fig. 1.

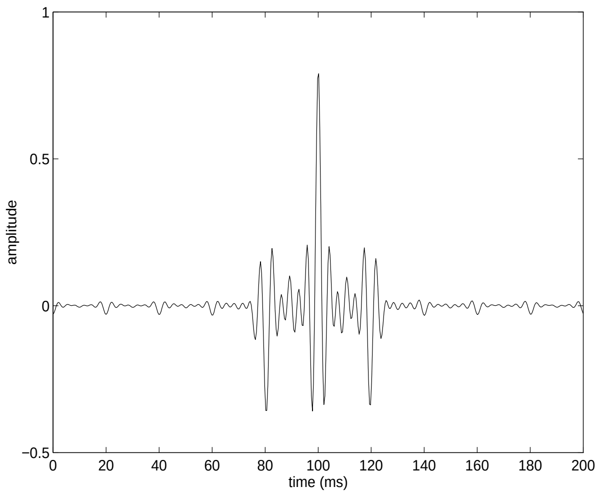

Fig. 3: Pulse generated with Ψ_1 as a hard constraint.

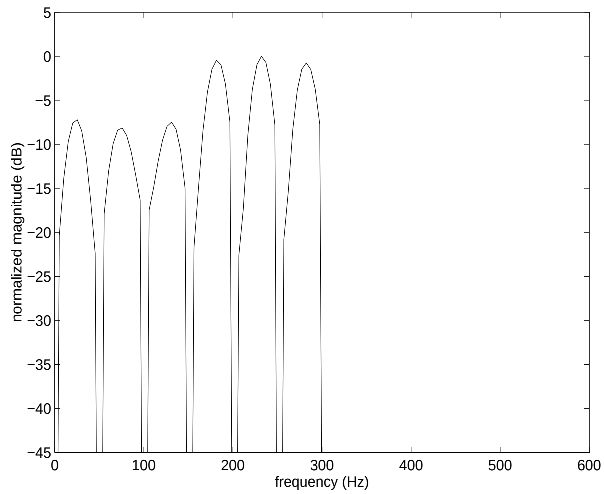

Fig. 4: Normalized spectral density of Fig. 3.

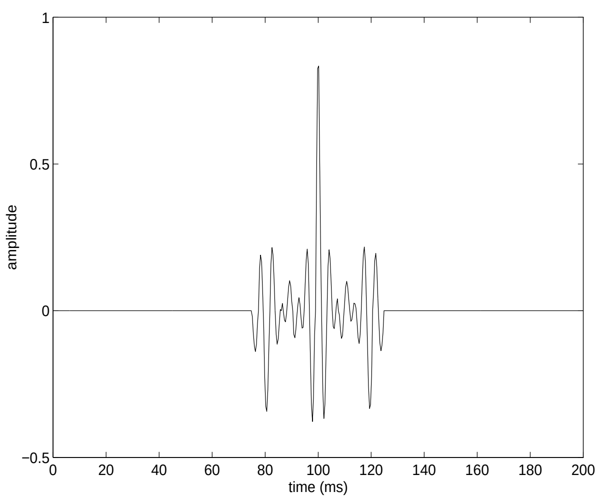

Fig. 5: Pulse generated with Ψ_4 as a hard constraint.

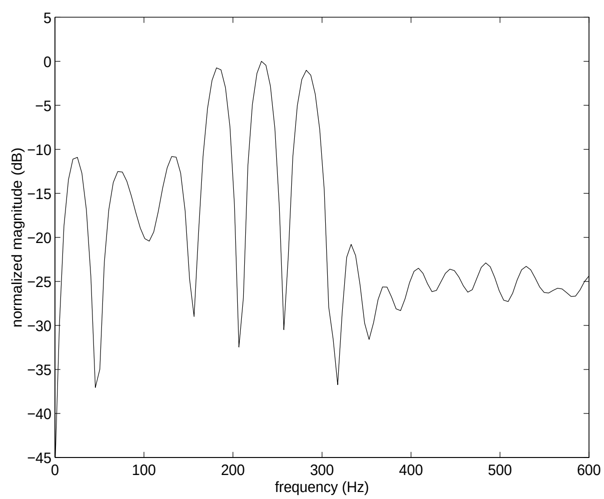

Fig. 6: Normalized spectral density of Fig. 5.

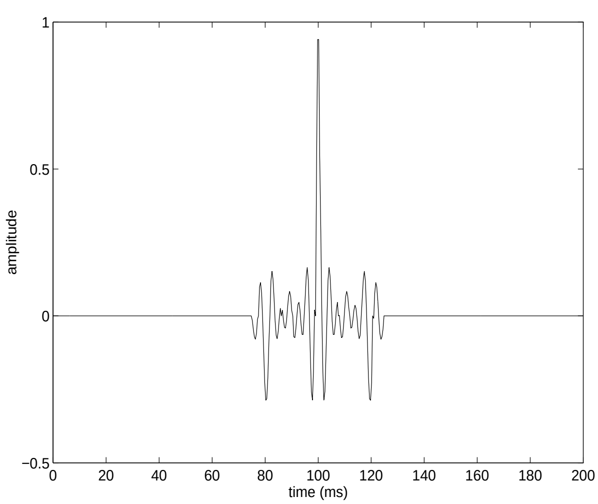

Fig. 7: Pulse generated by POCS.

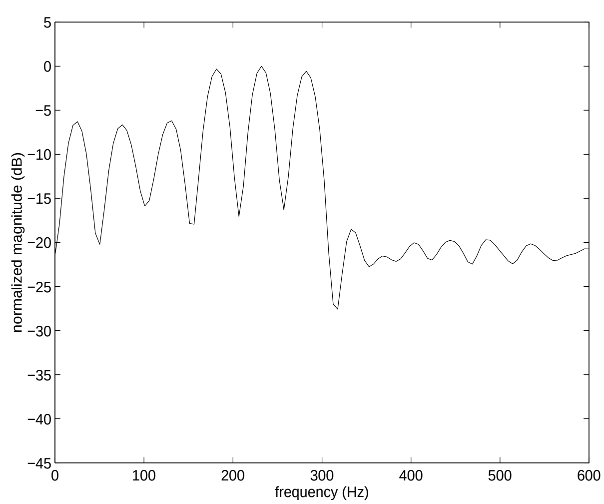

Fig. 8: Normalized spectral density of Fig. 7.