

THE CLOSEST-TO-MEAN FILTER: AN EDGE PRESERVING SMOOTHER FOR GAUSSIAN ENVIRONMENTS *

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ABSTRACT

Median based filters have gained wide-spread use because of their ability to preserve edges and suppress impulses. In this paper, we introduce the Closest-to-Mean (CTM) filter, which outputs the input sample closest to the sample mean. The CTM filtering framework offers lower computational complexity and better performance in near Gaussian environments than median filters. The formulation of the CTM is derived from the theory of S-filters, which form a class of generalized selection-type filters with the features of edge preservation and impulse suppression. S-filters can play a significant role in image processing, where edge and detail preservation are of paramount importance. We compare the performance of CTM, median, and mean filters in the smoothing of edges and impulses immersed in Gaussian noise. A sufficient condition for a signal to be a root of the CTM filter is included.

Data, figures and source code utilized in this paper are available at <http://www.ee.udel.edu/signals/robust>

Keywords - Selection filters, Closest-to-Mean filter, S-filters, selectification, edge preservation, root signals, locally monotonic signals, curvature, median filters.

I INTRODUCTION

Extracting signals from noisy data is a common problem faced in signal processing. In instances where contamination is additive and Gaussian, linear methods often provide the optimal tools. However, in the presence of signal edges and/or impulsive noise, the significant performance degradation presented by linear smoothers makes it necessary to resort to alternative nonlinear techniques.

Median filters have proven to be useful for smoothing applications in which edge preservation and impulse suppression are important requirements. Based on the nature of human visual perception, the above two properties are very important in image processing, where median based filters have gained overwhelming popularity [1].

Among the drawbacks of median filters, it is worth mentioning their lack of flexibility as well as their relatively high computational complexity¹. A further drawback of median based filters occurs in near Gaussian environments, where the sample median can lose as much as 40% efficiency compared to the sample mean [2].

As an alternative that overcomes the above limitations, this paper introduces the *Closest-to-Mean (CTM)* filter.

This new filter has its theoretical foundations based on the well known class of M-estimators of location [2, 3]. Using a procedure called *selectification*, an M-estimate can be restricted to be one of the input samples, providing it with the same *selection* property that median-based filters have [4].

The CTM filter is derived as the selection filter associated with the sample mean via the selectification procedure. In view of the optimality properties of the sample mean in Gaussian environments, the CTM filter becomes a very simple and efficient alternative for edge-preserving smoothing of discontinuous signals in Gaussian noise.

II SELECTIFICATION OF A LOCATION ESTIMATOR

Given a set of observations $S_x = \{x_1, x_2, \dots, x_N\}$ and a symmetric function $\rho(x)$ which is monotonic nondecreasing on $[0, \infty)$, the location M-estimate associated with ρ is defined as

$$\hat{\beta} = \arg \min_{\beta} \sum_{i=1}^N \rho(x_i - \beta), \quad (1)$$

where $\arg \min_{\beta}(\cdot)$ denotes the value of β that minimizes the expression inside the parenthesis [2, 3]. When $\rho(x) = x^2$, for example, the solution of (1) is given by the sample mean, whereas for $\rho(x) = |x|$, the solution is the sample median.

M-estimators are generalized forms of maximum-likelihood estimators for which $\rho(x) = -\log f(x)$, where $f(x)$ is the probability density of the additive noise within the samples. Depending on ρ , the minimization in (1) can be computationally complex, involving expensive iterative procedures. If we constrain the minimization space to be equal to the sample set S_x , the computational complexity of the problem can be easily reduced to $O(N^2)$. The output, $\hat{\beta}_s$, of such a constrained or “selectified” M-estimator is defined as

$$\hat{\beta}_s = \arg \min_{\beta \in S_x} \sum_{i=1}^N \rho(x_i - \beta). \quad (2)$$

These *selection type M-estimators* are referred to as S-estimators or S-filters [4]. In addition to the computational savings, the *selectification procedure provides M-estimators with the properties of edge preservation and impulse suppression exhibited by median filters* [4].

III THE CLOSEST-TO-MEAN FILTER

The simplest example of an S-estimate is generated from applying the selectification procedure to the sample mean. In this case $\rho(x) = x^2$, and the selectified mean, hereafter

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¹ The complexity associated with the computation of the sample median is, in general, on an order greater than or equal to $O(N \log N)$, where N is the sample size.

referred to as the *Closest-to-Mean (CTM)* estimator, is defined as:

$$\hat{\beta}_{CTM} = \arg \min_{x_j} \sum_{i=1}^N (x_i - x_j)^2. \quad (3)$$

More conveniently, this definition can be proven equivalent to

$$\hat{\beta}_{CTM} = \arg \min_{x_i} |x_i - \bar{x}|, \quad (4)$$

where $\bar{x}$ denotes the sample mean value. This formulation of the estimator, intuitively depicted in Fig. 1, results in an $O(N)$ complexity, as opposed to the $O(N^2)$ complexity associated with the definition in (3).

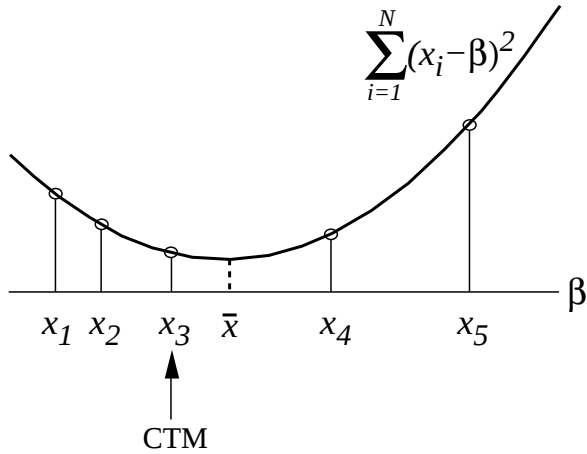


Figure 1. Determination of the CTM filter output. The CTM value corresponds to the input data which is closest to the sample mean.

IV ROOT SIGNALS ANALYSIS

An important problem for understanding the behavior of a selection filter is to characterize the signals that remain unchanged after the filter operation. The class of these signals, called *root signals*, has been extensively studied for the case of the median filter, where the characterization of the root signal set has played a fundamental role in the general understanding of the filter operation [1].

Characterizing the root signal set of the CTM filter is a very challenging problem, far from been complete. In this section, we characterize an important subclass of CTM root signals. We begin our work with a trivial proposition.

Proposition 1 *Straight lines are preserved after the operation of the CTM filter.*

In view of the above result, it is logical to ask for conditions under which signals with shapes “close” to a straight line would be preserved by the CTM filter. In order to answer this question, we first introduce the following definitions:

Definition 1 *Let $x(n)$ be a strictly monotonic signal. We define the instantaneous curvature of $x(n)$ as:*

$$\mathcal{C}(n) = \left| \frac{a_{n+1} - a_n}{a_n} \right|, \quad (5)$$

where a_n denotes the difference $x(n) - x(n-1)$. We refer to the non-negative value $\mathcal{C} = \sup_n \mathcal{C}(n)$ as the absolute curvature, or simply the curvature of $x(n)$.

It is easy to check that $\mathcal{C} = 0$, if and only if $x(n)$ is a nonconstant straight line. Hence, $\mathcal{C}$ can be seen as an indicator of how different from a straight line the shape of the signal is.

Definition 2 *The signal $x(n)$ is locally monotonic of size W if for any set of W consecutive points, the signal behaves monotonically, not necessarily strictly. In this case we denote $x(n)$ as being lomo(W) [1].*

Definition 3 *We refer to the signal $x(n)$ as second order locally monotonic of size W if it holds*

1. $x(n)$ is monotonic.
2. The difference signal $a(n) = x(n+1) - x(n)$ is locally monotonic of size W .

In this case we denote $x(n)$ as being lomo²(W).

Intuitively, lomo²(W) signals can be seen as ensembles of monotonic convex and monotonic concave pieces of length larger than or equal to W .

The following result, which we offer without proof, characterizes a subclass of root signals of the CTM filter:

Proposition 2 *Let $x(n)$ be a lomo²($2N+1$) signal with curvature*

$$\mathcal{C} < 1 - \left(\frac{N-1}{N+1} \right)^{\frac{1}{2N-1}}. \quad (6)$$

Then, $x(n)$ is a root signal of the CTM filter for any window length $W \leq 2N+1$.

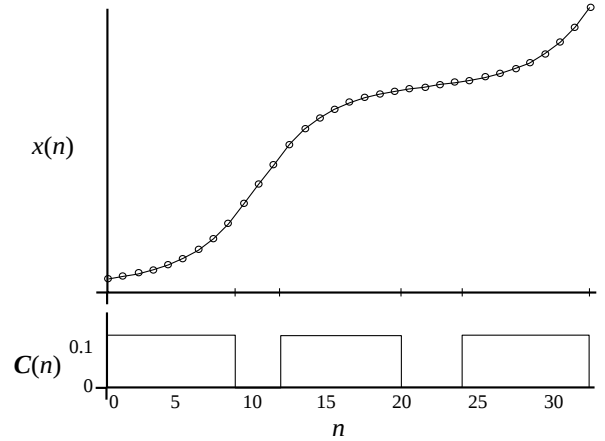


Figure 2. Typical “bounded curvature” CTM root signal as defined in Proposition 2. Top: Root signal $x(n)$. Bottom: Instantaneous curvature $\mathcal{C}(n)$. The zero curvature regions indicate that the signal is locally a straight line. (Window size $W = 7$).

V PROPERTIES AND APPLICATIONS

Although the exact theoretical analysis of the CTM filter performance looks difficult, it is easy to show that its variance as a constant-signal estimator is bounded by:

$$\text{VAR}(Z_{(1)}) \leq \text{VAR}(\hat{\beta}_{CTM}) \leq \text{VAR}(Z_{(N)}), \quad (7)$$

where $Z_{(i)}$ represents the i^{th} order statistic of the “centered” samples $Z_i = |x_i - \mu_x|$ (μ_x denotes the location parameter of the sample distribution).

The potential utility of the selectification procedure exploited by the CTM filter is illustrated in Fig. 3. In Fig. 3b, the mean filter has been used to smooth an image corrupted

by salt-and-pepper noise (Fig. 3a). The bad performance of the mean filter, indicated by both the smearing of edges and the poor job in removing the impulses, is a well documented fact in the image processing literature. Figure 3c shows the drastic performance improvement introduced by the selectification of the mean filter. The CTM-filtered image shows significant impulse suppression and at the same time maintains sharper edges than those in the mean-filtered image.



(a) Image of planes with salt-and-pepper noise



(b) Image after applying 3x3 mean filter



(c) Image after applying 3x3 CTM filter

Figure 3. Edge preservation and impulse suppression using the CTM filter.

A word of caution shall be stated here. Although the CTM filter can perform impulse suppression duties where the percentage of impulsive contamination is low, the CTM is not a robust estimator. Much as the mean filter does, the CTM presents a large breakdown point, and its performance can be significantly lower than other robust filters in impulsive environments. However, due to its intimate link with the mean filter, the CTM filter can play a significant role in the smoothing of discontinuous signals in additive Gaussian environments. Also, the impulse suppression capability makes CTM filtering appropriate for Gaussian noise signals smoothing in the presence of *low* impulsive contamination.

Figures 4 and 5 illustrate the performance of both CTM and median filters when smoothing impulses and edges in Gaussian noise. As it can be appreciated, both the median and the CTM filters perform the job of suppressing the

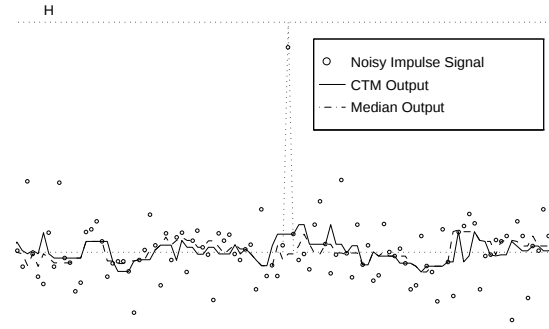


Figure 4. Smoothing of an impulse in Gaussian noise. (Window size $W = 7$).

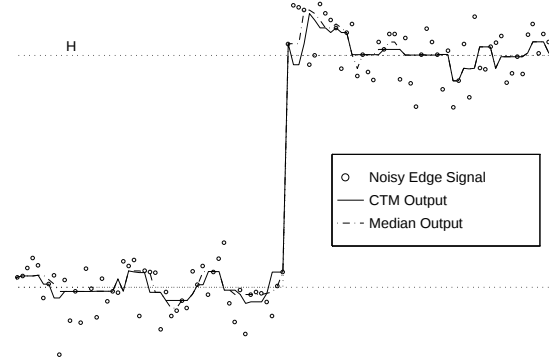


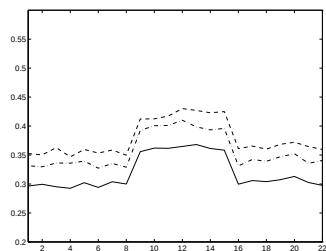
Figure 5. Smoothing of an edge signal in Gaussian noise. (Window size $W = 7$).

impulse, preserving the edge and smoothing the noise to acceptable levels. The CTM filter, however, presents much lower computational complexity than the median filter.

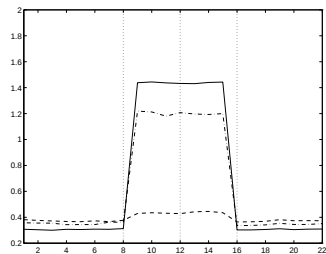
Figures 6 and 7 show Montecarlo-estimated mean absolute errors in the above filtering scenarios. Consistently for both impulse (Fig. 6) and edge (Fig. 7) signal smoothing, the CTM filter outperforms the median filter when the Gaussian variance is large relative to the edge height or the impulse magnitude. As the edge magnitude increases, the median operates at the lowest absolute error in the neighborhood of the edge, while the plots also show that, although the CTM is not the best, the value of its error is bounded. Out of the vicinity of either an edge or an impulse, the mean filter is the optimal smoother outperforming both CTM and median filters. The CTM filter, however, still shows better performance than the median filter.

VI CONCLUSIONS

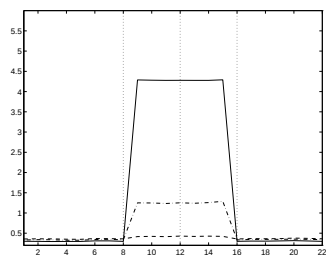
S-estimators are selectified versions of M-estimators that present the features of edge preservation and impulse suppression when used in a running window filter. The Closest-to-mean(CTM) filter is a novel selection-type filter derived from the theory of S-estimators. Its intimate link with the mean filter makes CTM filtering appropriate for denoising discontinuous signals in near Gaussian noise. We have analyzed several properties of this novel filter. Its performance, compared against the median filter, has been illustrated using Montecarlo simulations. It has been demonstrated that the CTM filter outperforms the median filter in edge preserving applications with large Gaussian noise variance. Some properties of the CTM filter have been addressed, including the characterization of a nontrivial class of the filter root signals. Due to its low complexity (on the order of $O(N)$), CTM filtering has a potential impact in applications where low computational cost is a paramount factor.



(a) $H/\sigma = 3$



(b) $H/\sigma = 10$



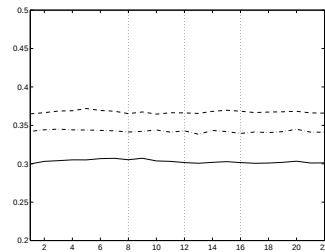
(c) $H/\sigma = 30$

Figure 6. Mean absolute error in the neighborhood of an impulse in Gaussian noise when smoothed with: (—) the mean filter, (---) the median filter, and (-.-) the CTM filter. H/σ denotes the ratio between the edge height and the noise standard deviation.

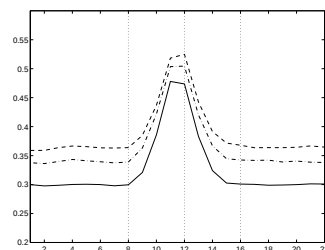
Further efforts are required in the study of the filter's statistical properties, and the characterization of the complete class of root signals. Generalizing the CTM filter to allow the introduction of weights is an interesting problem currently addressed by the authors. As illustrated in this paper, the selectification procedure could significantly improve the performance of linear FIR filters in image processing applications.

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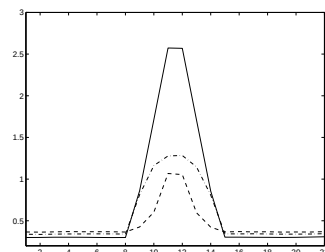
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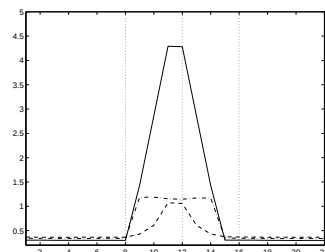
(a) $H/\sigma = 0$



(b) $H/\sigma = 1$



(c) $H/\sigma = 6$



(d) $H/\sigma = 10$

Figure 7. Mean absolute error in the neighborhood of an edge in Gaussian noise when smoothed with: (—) the mean filter, (---) the median filter, and (-.-) the CTM filter. H/σ denotes the ratio between the edge height and the noise standard deviation.