TOMOGRAPHY WITH UNKNOWN VIEW ANGLES

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ABSTRACT

We address the problem of parallel beam tomographic reconstruction when the angles at which the projections are taken are unkown. The problem arises in medical imaging owing to patient motion, and in imaging of viruses from a single projection of many identical units at random orientations. We determine conditions for unique identifiability of the angles from the projection data alone, and derive bounds on the variance of estimators of those angles in the presence of noise. Finally, we present a maximum likelihood estimator, along with a heuristic initialization procedure. Numerical simulations on a test phantom show excellent agreement with the bounds, and nearly perfect reconstructions at moderate noise levels.

1. INTRODUCTION

Much of the current research in tomography has focused on the problem of estimating an object from a finite collection of its line integral projections. It is generally assumed in such developments that the angles at which the projections are taken are known exactly. Yet there are instances of tomographic imaging in which perfect knowledge of the object's orientation is unobtainable. In medical imaging, for example, involuntary motion of the patient can result in uncertainty as to the data collection angles. The problem of reconstructing 3-D models of viruses from a single projection of many identical units at random orientations is another example in which the assumption of known orientation fails. In this paper, we examine the problem of determining the collection angles from the projections themselves. We derive conditions under which the angles can be uniquely recovered, and bounds on the variance of estimators of those angles in the presence of noise. Finally, we present some simulations, demonstrating the feasibility of the solution.

2. PROBLEM FORMULATION

For the purposes of this discussion, let us define the Radon transform operator parameterized by a vector $\boldsymbol{\theta} \in \Omega \triangleq [0,\pi]^P$ of angles as $\mathcal{R}_{\boldsymbol{\theta}} : L_2(\mathbb{D}^2) \to \{L_2(\mathbb{D})\}^P$, where $\mathbb{D}^2$ is the disk of unit radius in the plane, and $\mathbb{D}$ is the interval $[-1,1]$. The space $\{L_2(\mathbb{D})\}^P$ is the P-wise cartesian product of $L_2(\mathbb{D})$ with itself. Elements of this space are known as sinugrams, and are composed of P functions in $L_2(\mathbb{D})$ corresponding to the collection of all line integrals through an object in the direction perpendicular to θ_i:

$$\mathcal{R}_{\boldsymbol{\theta}} f(s,i) = \int f(s\cos\theta_i - t\sin\theta_i, s\sin\theta_i + t\cos\theta_i)\, dt \quad (1)$$

Our goal is to determine the parameter $\boldsymbol{\theta}$, which defines the collection angles, from the noisy sinugram. Specifically, we wish to determine $\boldsymbol{\theta}$ when the quantity measured is

$$\hat{g} \triangleq g + \nu = \mathcal{R}_{\boldsymbol{\theta}} f + \nu \quad (2)$$

where ν is additive white noise in each projection, and f is the unknown object being imaged. We also assume that the noise is uncorrelated across projections. Once the angles have been determined, standard reconstruction techniques can be applied to the projections to estimate f. To avoid trivial degeneracies, we assume that the acquisition angles are all distinct, i.e, that $\theta_i \neq \theta_j$ for $i \neq j$.

3. EXISTENCE AND UNIQUENESS

Our primary tools in exploring the identifiability of the angles are the Helgasson-Ludwig (HL) consistency conditions, which characterize elements in the range space of $\mathcal{R}_{\boldsymbol{\theta}}$ in terms of their geometric moments and $\boldsymbol{\theta}$ [1]. Given a sinugram $g \in \{L_2(\mathbb{D})\}^P$, the parameterized operator $\mathcal{M}_k : \{L_2(\mathbb{D})\}^P \to \mathbb{R}^P$ returns the kth geometric moment of each projection as a vector in $\mathbb{R}^P$ for $k \geq 0$:

$$\mathcal{M}_k g(i) = \sqrt{\frac{2k+1}{2}} \int_{\mathbb{D}} s^k g(s,i)\, ds \quad i = 1, \ldots, P. \quad (3)$$

We will also need moments of the object $f \in L_2(\mathbb{D}^2)$ being imaged. These moments are generated by a family of linear operators $\mathcal{G}_k : L_2(\mathbb{D}^2) \to \mathbb{R}^{k+1}$, which for a given $k \geq 0$, return all geometric moments of f of total order k:

$$\mathcal{G}_k f(i) = \iint_{\mathbb{D}^2} x^{i-1} y^{k-i+1} f(x,y)\, dx\, dy \quad i = 1, \ldots, k+1. \quad (4)$$

The following is a necessary form of the HL conditions, specialized to parallel beam geometry, and with the dependency on $\boldsymbol{\theta}$ and the moments of f made explicit:

This work was supported in part by National Science Foundation grant no. MIP-9157377 and by a graduate fellowship from the Joint Services Electronics Program under grant no. F49620-96-1-0216.

Theorem 1 (Helgasson-Ludwig) *If $g = \mathcal{R}_{\boldsymbol{\theta}} f$ for some $\boldsymbol{\theta} \in \Omega$, $f \in L_2(\mathbb{D}^2)$, then for $k \geq 0$, $i = 1, \ldots, P$*

$$\mathcal{M}_k g(i) = \sqrt{\frac{2k+1}{2}} \sum_{j=0}^{k} \binom{k}{j} \mathcal{G}_k f(j+1) \cos^j \theta_i \sin^{k-j} \theta_i. \tag{5}$$

Hence, the HL conditions, in the absence of *a priori* information on f (except, perhaps, that it lies in $L_2(\mathbb{D}^2)$), yield an explicit relationship between the object moments, the collection angles $\boldsymbol{\theta}$, and the measurable projection moments. We outline conditions for unique determination of $\boldsymbol{\theta}$ and the object moments by examining the solutions to (5) for multiple k, where the object moments and the collection angles are treated as unknowns.

Two trivial types of non-uniqueness arise immediately from the formulation of problem. The first type results from a lack of an absolute frame of reference; for a given image, any rotation of the image produces the same sinugram if the collection angles are similarly rotated. Thus, we can determine the angles of data collection to within a rotation only. The second type of indeterminacy results from reflecting all angles about any one projection. For example, if we estimate $\hat{\boldsymbol{\theta}} = -\boldsymbol{\theta}$, we still get a consistent sinugram, but with the object similarly reflected. We must therefore extend our notion of "unique solution" to include these types of indeterminacies. For any set of collection angles, we define an equivalence class of solutions:

$$S(\boldsymbol{\theta}) = \{\boldsymbol{\phi} \in \Omega \mid [\boldsymbol{\phi} + c]_\pi = \pm\boldsymbol{\theta}, c \in \mathbb{R}\} \tag{6}$$

where $[\cdot]_\pi$ denotes modulo π.

Consider now the problem of determining the angles from projection moments of order m through n, to which we refer as an (m, n) method. An (m, n) method attempts to jointly solve the system of $n - m + 1$ HL equations for order $k = m, \ldots, n$, where the unknowns are the P projection angles and the object moments of total order m through n. With the equivalence class defined as above, we have:

Theorem 2 (Necessary Condition [2]) *A unique solution to within an equivalence class for an (m, n) method exists only if*

$$P \geq \frac{(m+n+2)(n-m+1)}{2(n-m)} \qquad n > m > 0. \tag{7}$$

Theorem 3 (Sufficient Conditions for $(1,2)$ [2]) *If $P > 8$, $\|\mathcal{M}_2(g)\| \neq 0$, $\|\mathcal{M}_1(g)\| \neq 0$, and $\nexists \alpha \in \mathbb{R}$ such that $\mathcal{M}_2 g(i) = \alpha \mathcal{M}_1 g(i)^2$ for all i, then there exists a unique solution to within an equivalence class for the $(1,2)$ equations.*

Theorem 4 (Sufficient Conditions for $(2,3)$ [2]) *If $P > 24$, $\|\mathcal{M}_3(g)\| \neq 0$, $\|\mathcal{M}_2(g)\| \neq 0$, and $\nexists \alpha, \omega_1, \omega_2 \in \mathbb{R}$ such that*

$$\mathcal{G}_2 f = \alpha[\omega_1^2, \omega_1\omega_2, \omega_2^2]^T \tag{8}$$

then there exists a unique solution to within an equivalence class for the $(2,3)$ equations.

The conditions on the moments are degeneracy conditions that will be satisfied for generic images. Since both Theorem 3 and Theorem 4 relate to second order moment information, a candidate object can be tested for these degeneracies by replacing it with an ellipse that matches its first and second moments. Theorem 3 states that uniqueness of the $(1,2)$ method breaks down when the line through the origin and the center of the ellipse is colinear with either the major or minor ellipse axis. Theorem 4 states that uniqueness of the $(2,3)$ method breaks down when the image degenerates to a line distribution (i.e., it is completely correlated in some direction). Uniqueness of solutions to the $(2,3)$ method implies that even if the first moment information has been lost (e.g. in the process of aligning the projections), the angles can still be recovered to within a rotation and/or reflection.

4. PERFORMANCE BOUNDS - MOMENT-BASED METHODS

We derive bounds on the performance of estimators that jointly estimate the projection angles and the object moments, in the presence of zero mean additive white Gaussian noise in the projections.

The measured moments become Gaussian random variables:

$$\hat{\mu}_k(i) = \mathcal{M}_k \hat{g}(i) = \mathcal{M}_k g(i) + \mathcal{M}_k \nu(i). \tag{9}$$

Under the assumption that the noise is uncorrelated between projections, and has power spectral density N_0 the following properties then result:

$$\begin{aligned}
\mathrm{E}\{\hat{\mu}_k(i)\} &= \mathcal{M}_k g(i) && (10)\\
\mathrm{var}(\hat{\mu}_k) &= N_0 && (11)\\
\mathrm{cov}(\hat{\mu}_k(i), \hat{\mu}_l(j)) &= 0 \quad i \neq j && (12)\\
\mathrm{cov}(\hat{\mu}_k(i), \hat{\mu}_l(i)) &\triangleq \sigma_{kl} && (13)\\
&= \begin{cases} N_0 \frac{\sqrt{(2k+1)(2l+1)}}{k+l+1} & k+l+1 \text{ odd} \\ 0 & \text{else.} \end{cases}
\end{aligned}$$

Consider now the estimation of the object moments of total order m through n, jointly with the projection angles. These unknowns can be concatenated into a vector:

$$\boldsymbol{x} = \left[\mathcal{G}_m f^T, \mathcal{G}_{m+1} f^T, \cdots, \mathcal{G}_n f^T \boldsymbol{\theta}^T\right]^T. \tag{14}$$

Since the number of object moments to be determined is $\frac{1}{2}(n-m+1)(n+m+2) \triangleq \eta(m,n)$, $\boldsymbol{x}$ is a vector in $\mathbb{R}^{\eta(m,n)+P}$. The measurements are the noisy moments of the projections, and are stacked by order:

$$h(\boldsymbol{x}) = \left[\hat{\mu}_m^T, \hat{\mu}_{m+1}^T, \cdots, \hat{\mu}_n^T\right]^T \tag{15}$$

such that for each $\boldsymbol{x}$, $h(\boldsymbol{x}) \in \mathbb{R}^{(n-m+1)P}$. The Fischer information matrix can then be written as $\boldsymbol{H}\boldsymbol{K}^{-1}\boldsymbol{H}^T$ where $\boldsymbol{H}$ is an $\eta(m,n) + P \times (n-m+1)P$ matrix with the following structure:

$$\boldsymbol{H} = \nabla_{\boldsymbol{x}} h^T(\boldsymbol{x}) = \begin{bmatrix} \boldsymbol{T}_m & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{T}_{m+1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{T}_n \\ \boldsymbol{D}_m & \boldsymbol{D}_{m+1} & \cdots & \boldsymbol{D}_n \end{bmatrix} \quad (16)$$

where $\boldsymbol{T}_i$ is a matrix of size $i+1 \times P$, and the jth row of $\boldsymbol{T}_i$ is $\sqrt{\frac{2i+1}{2}}\binom{i}{j-1}\cos^{j-1}\boldsymbol{\theta}\sin^{i-j+1}\boldsymbol{\theta}$ for $j = 1, \ldots, i+1$. The diagonal matrix $\boldsymbol{D}_i$ is size $P \times P$, and the diagonal elements are

$$(\boldsymbol{D}_i)_{l,l} = \sqrt{\frac{2i+1}{2}} \sum_{j=0}^{i} \binom{i}{j} \mathcal{G}_i f(j+1)[(i-j)\cos^{j+1}\theta_l \times \sin^{i-j-1}\theta_l - j\cos^{j-1}\theta_l \sin^{i-j+1}\theta_l]. \quad (17)$$

Due to the nature of the noise model proposed, the inverse noise covariance matrix $\boldsymbol{K}$ can be written as $\boldsymbol{Q}^{-1} \otimes \boldsymbol{I}_P$, where $\boldsymbol{I}_P$ is the $P \times P$ identity matrix, and $(\boldsymbol{Q})_{i,j} = \sigma_{ij}$ defined in (14).

The presence of the rotational uncertainty in the angles implies that the Fischer information matrix will have at most rank $(m-n)P$. This can be handled by setting $\theta_1 = 0$, and thus fixing the reference frame. The row and column of the FIM corresponding to θ_1 can then be deleted. Cramér-Rao bounds on the variance of unbiased estimators can then be computed for any given scenario defined by object moments and the collection angles.

An important point to note is the behavior of the FIM when $n \geq P$. In such a case, uniqueness of the object moment estimates breaks down since there are more unknown object moments of total order n, than equations (one equation for each of the P projections). Furthermore, we can show the following upper bound on the number of moments useful in estimating $\boldsymbol{\theta}$:

Theorem 5 *[2] No moment-based estimator of $\boldsymbol{\theta}$ can improve by the addition of the kth moment equations for $k \geq P$.*

5. SIMULATIONS

It is not the intent of this discussion to provide optimal algorithms for the angle estimation problem, but rather to demonstrate feasibility using a phantom and the maximum likelihood estimate of the angles. The form of the MLE used is most easily derived by writing the HL equations in matrix form:

$$\boldsymbol{y} = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{s} + \boldsymbol{n} \quad (18)$$

where $\boldsymbol{y} = [\mathcal{M}_m^T g, \mathcal{M}_{m+1}^T g, \ldots, \mathcal{M}_n^T g]^T \in \mathbb{R}^{(n-m+1)P}$, $\boldsymbol{s} = [\mathcal{G}_m^T f, \mathcal{G}_{m+1}^T f, \ldots, \mathcal{G}_n^T f]^T \in \mathbb{R}^{\eta(m,n)}$, and $\boldsymbol{A}(\boldsymbol{\theta})$ is a suitably defined matrix function of $\boldsymbol{\theta}$. With the geometric moment formulation, recall that the noise vector $\boldsymbol{n}$ is not white. To diagonalize the noise covariance, we apply a simple linear whitening transformation, henceforth implicit in the notation. Minimizing (18) over $\boldsymbol{s}$ for fixed $\boldsymbol{\theta}$ yields

$$\hat{\boldsymbol{s}} = \boldsymbol{A}(\boldsymbol{\theta})^{\dagger}\boldsymbol{y}, \quad (19)$$

which upon substitution yields the following non-linear least squares problem:

$$\hat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \|\boldsymbol{P}_{R(\boldsymbol{A}(\boldsymbol{\theta}))^{\perp}}\boldsymbol{y}\|^2. \quad (20)$$

The phantom is composed of ten disks of different radius and uniform density shown in Figure 3a. Cramér-Rao bounds on the performance of $(1, k)$ methods for a number of different k are plotted in Figure 1, for 75 random angles chosen to yield approximately uniform coverage of $[0, \pi)$. The noise intensity N_0 was -30dB. To demonstrate that these bounds are at least locally accurate, we approximate the MLE via a pseudo-newton type descent algorithm applied to (20) for a variety of noise levels and k. Figure 2 depicts the results of the Monte Carlo runs, averaged over 100 noise realizations per data point, and with a fixed initial guess chosen close to the global minimum. Even at extremely high noise levels, the MLE does quite well, relative to the Cramér-Rao bounds.

Although informative, this performace relies on finding an initial guess close to the global minimum. To demonstrate that good initial guesses can be found for $N_0 \leq -25$dB, we outline the following heuristic algorithm [2], designed for full angle tomographic imaging problems with a reasonably large number of projections.

1. Identify two projections with smallest and largest $|\hat{\mu}_1|$ and assign these projections to 0 and 90 degrees.
2. Solve the first order moment equation to determine the two candidate angles for each of the remaining projections, and decide which candidate is correct using the second moment information.

Empirically, this algorithm works well for $N_0 \leq -25$dB. Once the initial guess is generated, the MLE is approximated by the BFGS pseudo-newton method (see, e.g. [3]) initialized with the output of this hueristic algorithm.

This approach to calculating the MLE was tested on phantom of Figure 3, at the same set of 75 angles used in the Cramér-Rao bound studies. The algorithm was tested at $N_0 = -30$dB, and for this phantom, also works for slightly higher noise levels. Noiseless and noisey filtered back projections (FBP) at the correct angles are depicted in Figures 3a and 3b respectively. If the angles are assumed to be uniformly spaced and ordered in increasing angle, the reconstruction is useless, as evidenced by Figure 3f. Such an assumption may be a valid starting point in the case that the projections are collected sequentially and the uncertainty in the angles is small. For the randomly ordered angles used in this experiment, however, such an assumption is clearly unjustified.

To isolate the effects of noise in the projections from the effects of uncertainties in the angles, the FBP reconstructions at the initial guess angles (Figure 3e), and the MLE angles (Figure 3c) are done with the noiseless projections (but with angles estimated from the noisy data). The distortions present in Figure 3e are then clearly visible, while the MLE yields a reconstruction that is almost indistinguishable from Figure 3a. For the sake of comparison, the FBP reconstruction with the actual noisy projections used to estimate the angles is depicted in Figure 3d.

6. CONCLUSION

We have examined the problem of tomographic reconstruction in the case of unknown angles. Except under certain degenerate conditions, the angles can be determined to within a rotation and/or reflection given enough projections. Furthermore, bounds on the variance of moment-based angle estimates can be computed, and do not improve when moments of order greater than P, the number of projections, are added. Simulations with a disk phantom demonstrate that the bounds are accurate, and can be approached by the MLE for sufficiently good initial guesses. However, global analytical bounds on estimator performance would be more generally useful. We have presented a hueristic initialization algorithm which works well under certain conditions (i.e. a reasonably large number of angles, and low noise levels) merely to demonstrate feasibility. Future work will focus on efficient and global methods for estimation.

7. REFERENCES

[1] F. Natterer, *The Mathematics of Computerized Tomography.* New York: Wiley, 1986.

[2] S. Basu, "2D Parallel Beam Tomography with Uncertain Geometry," M.S. Thesis, Univ. of Ill. at Urbana-Champaign, Urbana, IL, 1997.

[3] D. P. Bertsekas, *Non-linear Programming.* Belmont, Mass. : Athena Scientific, 1995.

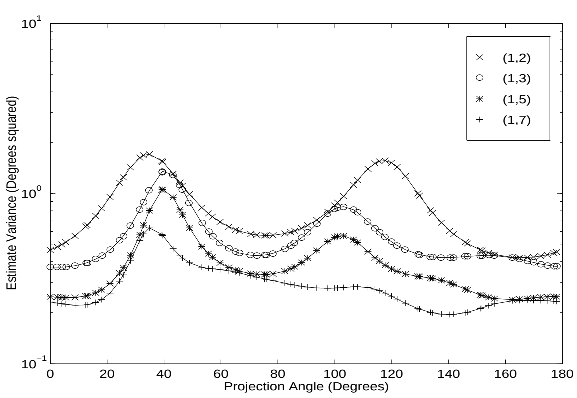

Figure 1: Cramér-Rao bounds for the disk phantom at $N_0 = -30$dB for 75 angles (the symbols mark the angles used in the calculations)

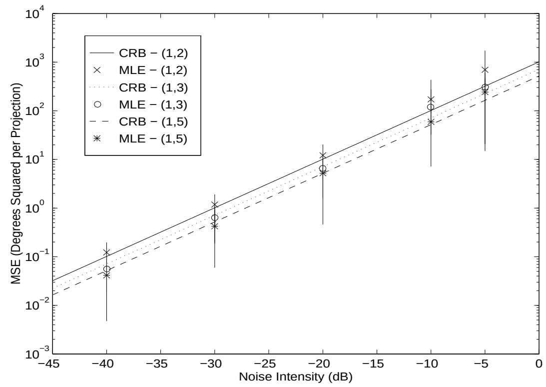

Figure 2: Monte Carlo simulations of the MLE for $(1, k)$ methods.

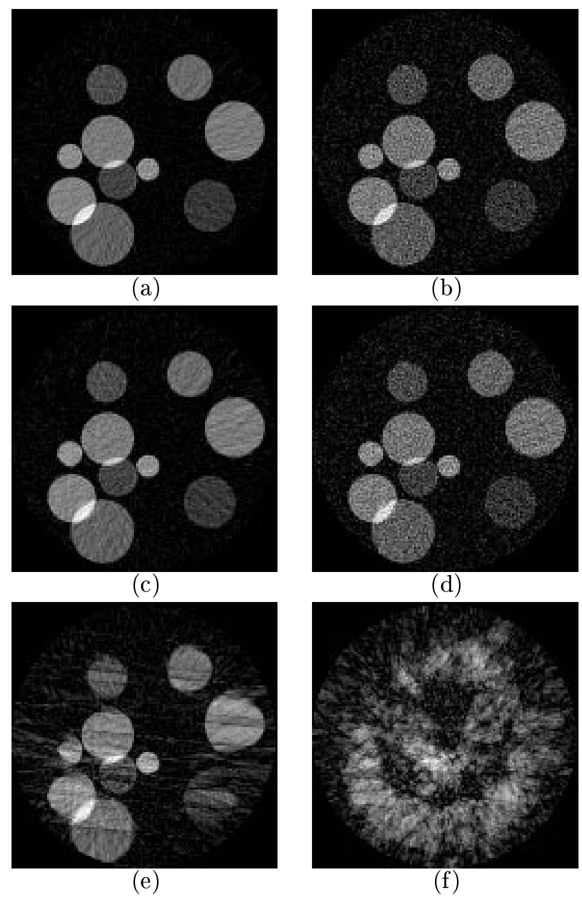

Figure 3: (a) FBP of noiseless projections at correct angles. (b) FBP of noisy projections at correct angles with $N_0 = -30$dB. (c) FBP of noiseless projections at MLE angles. (d) FBP of noisy projections at MLE angles. (e) FBP of noiseless projections at initial guess angles (generated by hueristic algorithm). (f) FBP of noiseless projections where angles are assumed correctly ordered and uniformly spaced.