PARAMETER ESTIMATION FOR LINEAR MULTICHANNEL MULTIDIMENSIONAL MODELS OF NON-GAUSSIAN DISCRETE RANDOM FIELDS

Jitendra K. Tugnait
Dept. of Electrical Engineering
Auburn University, Auburn, Alabama 36849, USA
tugnait@eng.auburn.edu

ABSTRACT

This paper is concerned with the problem of estimating the multichannel impulse response function of a 2-D multiple-input multiple-output (MIMO) system given only the measurements of the vector output of the system. Such models arise in a variety of situations such as color images (textures), or image data from multiple frequency bands, multiple sensors or multiple time frames. We extend the approach of Tugnait(1994) (which deals with SISO 2-D systems) to MIMO 2-D systems. The paper is focused on certain theoretical aspects of the problem: estimation criteria, existence of a solution, and parameter identifiability. An iterative, inverse filter criteria based approach is developed using the third-order and/or fourth-order normalized cumulants of the inverse filtered data at zero-lag. The approach is input-iterative, i.e., the input sequences are extracted and removed one-by-one. The matrix impulse response is then obtained by cross-correlating the extracted inputs with the observed outputs.

1 Introduction

This paper is concerned with the problem of estimating the multichannel impulse response function of a 2-D multiple-input multiple-output (MIMO) system given only the measurements of the vector output of the system. Such (multichannel multidimensional) models arise in a variety of situations such as color images (textures), or image data from multiple frequency bands, multiple sensors or multiple time frames; see [5] and [6], and references therein.

Linear parametric models for multidimensional random signals have been found useful in many applications such as image coding, enhancement, restoration, synthesis, classification, and spectral estimation [7]. A vast majority of this work has concentrated on exploitation of only the second-order statistics of the data either explicitly by restricting attention to the correlation properties of the multidimensional signal, or implicitly by assuming that the signal is Gaussian. A consequence of this is that either the underlying models should be quarter-plane (or, half-plane) causal and minimum phase, or the impulse response of the underlying parametric model must possess certain symmetry (such as "symmetric noncausality"), in order to achieve parameter identifiability [2].

Recently ([1]-[3]) it has been shown that higher-order cumulant functions of the underlying random field can be exploited (in addition to or in lieu of the usual second-order statistics) to fit more general "phase-sensitive" models where one captures both the system transfer function magnitude as well as the system transfer function phase unlike the second order statistics (mean and correlation functions) case which depends only upon the system transfer function magnitude. Possible advantages of such modeling in the context of monochrome texture synthesis using single-channel 2-D model fitting has been demonstrated in [2] and [3]. In this paper we extend the approach of [2] (which deal with SISO 2-D systems) to MIMO 2-D systems. The paper is focused on certain theoretical aspects of the problem: estimation criteria, existence of solution, parameter identifiability etc. It is an extension of our recent results on 1-D MIMO systems reported in [4]. An iterative, inverse filter criteria based approach is developed using the third-order and/or fourth-order normalized cumulants of the inverse filtered data at zero-lag. The approach is input-iterative, i.e., the input sequences are extracted and removed one-by-one. The matrix impulse response is then obtained by cross-correlating the extracted inputs with the observed outputs.

2 Model Assumptions

Consider a 2-D MIMO system with N outputs and M inputs. The i-th component of the output at pixel position (m,n) is given by $(i = 1, 2, \cdots, N)$

$$y_i(m,n) = \sum_{j=1}^{M} F_{ij}(z_1,z_2)w_j(m,n) + v_i(m,n), \tag{2-1}$$

$$\Rightarrow \quad \mathbf{y}(m,n) = \mathbf{F}(z_1,z_2)\mathbf{w}(m,n) + \mathbf{v}(m,n), \tag{2-2}$$

where $\mathbf{y}(m,n) = [y_1(m,n) \vdots y_2(m,n) \vdots \cdots \vdots y_N(m,n)]^T$, similarly for $\mathbf{w}(m,n)$ and $\mathbf{v}(m,n)$, z_1 is both, the backward-shift operator (i.e., $z_1^{-1}w(m,n) = w(m-1,n)$, etc.) and the complex-variable in the 2-D $\mathcal{Z}$-transform, $w_j(m,n)$ is the j-th input at pixel position

This work was supported by the National Science Foundation under Grant MIP-9312559.

(m,n), $y_i(m,n)$ is the i-th output, $v_i(m,n)$ is the additive Gaussian measurement noise, and

$$F_{ij}(z_1,z_2) \;:=\; \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f_{ij}(k,l)z_1^{-k}z_2^{-l} \quad (2-3)$$

is the scalar transfer function with $w_j(m,n)$ as the input and $y_i(m,n)$ as the output.

The following assumptions are made (more restrictions are imposed later in Sec. 3.1):

- **(AS1)** The vector 2-D sequence $\mathbf{w}(m,n)$ is assumed to be zero-mean and i.i.d. both componentwise and spatially. Also assume that fourth-cumulant or the third-cumulant of $\mathbf{w}(m,n)$ is nonzero.
- **(AS2)** If it is an infinite impulse response (IIR) model, then (2-2) is assumed to be the result of a finite-dimensional multichannel multidimensional ARMA model such that the model matrix impulse response function is exponentially stable. (See also [2].)

3 Estimation Criteria

We assume in the rest of this section that the noise $\mathbf{v}(m,n)$ in (2-2) is negligible. Let $\mathrm{CUM}_r(w)$ denote the r-th (r=3 or 4) order cumulant of a zero-mean random variable w, defined as

$$\mathrm{CUM}_4(w) \;=\; E\{w^4\} - 3[E\{w^2\}]^2, \qquad (3-1)$$

$$\mathrm{CUM}_3(w) \;=\; E\{w^3\}; \quad \mathrm{CUM}_2(w) \;=\; E\{w^2\}. \qquad (3-2)$$

We will use the notation $\gamma_{rwi} = \mathrm{CUM}_r(w_i(m,n))$ and $\sigma_{wi}^2 = E\{|w_i(m,n)|^2\}$. Consider an $1 \times N$ row-vector polynomial equalizer $\mathbf{C}^T(z_1,z_2)$, with its j-th entry denoted by $C_j(z_1,z_2)$, operating on the data vector $\mathbf{y}(m,n)$ (see (2-2)). Let the equalizer output be denoted by $e(m,n)$. We then have

$$e(m,n) = \sum_{i=1}^{N} C_i(z_1,z_2)y_i(m,n) = \sum_{j=1}^{M} H_j(z_1,z_2)w_j(m,n) \qquad (3-3)$$

where

$$C_j(z_1,z_2) \;=\; \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} c_j(k,l)z_1^{-k}z_2^{-l}, \quad (3-4)$$

$$H_j(z_1,z_2) \;:=\; \sum_{i=1}^{N} C_i(z_1,z_2)F_{ij}(z_1,z_2). \quad (3-5)$$

In general, we have

$$H_j(z_1,z_2) \;=\; \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} h_j(k,l)z_1^{-k}z_2^{-l}. \quad (3-6)$$

Define $\overline{h}_j(m,n) = \sigma_{wj}h_j(m,n)$, $\overline{\gamma}_{rwj} = \gamma_{rwj}/\sigma_{wj}^4$ and $|\overline{\gamma}_{rmax}| := \max_{1\le j\le M}|\overline{\gamma}_{rj}|$. It therefore follows that

$$\mathrm{CUM}_r(e(m,n)) \;=\; \sum_{j=1}^{M}\overline{\gamma}_{rwj}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\overline{h}_j^{\,r}(k,l) \qquad (3-7)$$

It is easy to see that $\overline{\gamma}_{2wj} = 1$ for all j.

Consider the family of cost functions

$$J_{r2}(\mathbf{c}) \;:=\; |\mathrm{CUM}_r(e(m,n))| \,/\, |\mathrm{CUM}_2(e(m,n))|^{r/2} \qquad (3-8)$$

where r is a positive integer such that either r=3 or r=4. We have from (3-7)

$$|\mathrm{CUM}_3(e(m,n))| \;\le\; |\sum_{j=1}^{M}\overline{\gamma}_{3wj}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\overline{h}_j^{3}(k,l)|$$

$$\le\; |\overline{\gamma}_{3max}|\sum_{j=1}^{M}\frac{|\overline{\gamma}_{3wj}|}{|\overline{\gamma}_{3max}|}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}|\overline{h}_j(k,l)|^3 \qquad (3-9)$$

where $|\overline{\gamma}_{3max}| = \max_{1\le j\le M}\,|\overline{\gamma}_{3wj}|$. It therefore follows that

$$|\mathrm{CUM}_3(e(m,n))| \;\le\; |\overline{\gamma}_{3max}|\sum_{j=1}^{M}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}|\overline{h}_j(k,l)|^3$$

$$\le\; |\overline{\gamma}_{3max}|\left[\sum_{j=1}^{M}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}|\overline{h}_j(k,l)|^2\right]^{1.5}. \qquad (3-10)$$

Using (3-7), (3-8), (3-10) and the fact that $\overline{\gamma}_{2wj} = 1\ \forall j$, we have

$$J_{32}(\mathbf{c}) \;\le\; |\overline{\gamma}_{3max}|. \qquad (3-11)$$

Now $\sum_j\sum_k\sum_l \overline{h}_j^4(k,l) \;=\; [\sum_j\sum_k\sum_l \overline{h}_j^2(k,l)]^2$ if and only if ($j_0 \in \{1,2,\cdots,M\}$)

$$\overline{h}_j(k,l) \;=\; d\delta(k-k_0)\delta(l-l_0)\delta(j-j_0), \quad (3-12)$$

where d is some constant, k_0 and l_0 are some integers, j_0 indexes some input out of the given M inputs, and $\delta(k-k_0) = 1$ if $k = k_0$, $= 0$ otherwise. It therefore follows that (3-11) holds true with equality if and only if (3-12) is true with the exception that now j_0 should be such that $|\overline{\gamma}_{3wj_0}| = |\overline{\gamma}_{3max}|$.

Mimicking the above arguments it is also easy to establish that

$$J_{42}(\mathbf{c}) \;\le\; |\overline{\gamma}_{4max}| \qquad (3-13)$$

with equality if and only if (3-12) is true with the exception that now j_0 should be such that $|\overline{\gamma}_{4wj_0}| = |\overline{\gamma}_{4max}| := \max_{1\le j\le M}\,|\overline{\gamma}_{4wj}|$.

Thus, when (3-12) holds, (3-3) reduces to

$$e(m,n) \;=\; dw_{j_0}(m-k_0,n-l_0). \qquad (3-14)$$

3.1 Does such a solution exist?

It follows from (3-6) and (3-12) that

$$H_j(z_1,z_2) \;=\; dz_1^{-m_0}z_2^{-n_0}\delta(j-j_0) \qquad (3-15)$$

which when combined with (3-5), yields ($j = 1,2,\cdots,M$)

$$\sum_{i=1}^{N} C_i(z_1,z_2)F_{ij}(z_1,z_2) \;=\; \begin{cases} dz_1^{-m_0}z_2^{-n_0} & \text{if} \quad j=j_0 \\ 0 & \text{if} \quad j\neq j_0. \end{cases} \qquad (3-16)$$

From (3-16) we have the matrix polynomial equation

$$\mathbf{F}^T(z_1,z_2)\mathbf{C}(z_1,z_2) \;=\; \mathbf{E}(z_1,z_2) \qquad (3-17)$$

where

$$\mathbf{F}(z_1,z_2) = \begin{bmatrix} F_{11}(z_1,z_2) & \cdots & F_{1M}(z_1,z_2) \\ F_{21}(z_1,z_2) & \cdots & F_{2M}(z_1,z_2) \\ \vdots & \cdots & \vdots \\ F_{N1}(z_1,z_2) & \cdots & F_{NM}(z_1,z_2) \end{bmatrix}, \qquad (3-18)$$

and

$$\mathbf{E}(z_1,z_2) = [\,0 \;\; 0 \;\; \cdots \;\; 0 \;\; dz_1^{-m_0}z_2^{-n_0} \;\; 0 \;\; \cdots \;\; 0\,]^T. \qquad (3-19)$$

Note that the $M-$column vector $\mathbf{E}(z_1,z_2)$ has the nonzero entry in its j_0-th row with zeros every place else, and $\mathbf{F}(z_1,z_2)$ is $N\times M$ and $\mathbf{C}(z_1,z_2)$ is $N\times 1$.

A set of sufficient conditions for the existence of the desired solution can be easily deduced from (3-17). These are

- **(SC1)** $N \geq M$, i.e. at least as many outputs as inputs.
- **(SC2)** Rank$\{\mathbf{F}(z_1,z_2)\} = M$ for any $|z_1| = 1 = |z_2|$.

Consider the solution (* denotes complex conjugation)

$$\overline{\mathbf{C}}(z_1,z_2) =$$

$$\mathbf{F}^*(1/z_1^*,1/z_2^*)[\mathbf{F}^T(z_1,z_2)\mathbf{F}^*(1/z_1^*,1/z_2^*)]^{-1}\mathbf{E}(z_1,z_2). \qquad (3-20)$$

By Result A.18.2 on p. 655 of [8] and **(SC2)**, the inverse in (3-20) exists for any $|z_1| = 1 = |z_2|$. Equivalently, $\det\big(\mathbf{F}^T(z_1,z_2)\mathbf{F}^*(1/z_1^*,1/z_2^*)\big) \neq 0$ for any $|z_1| = 1 = |z_2|$ which combined with **(AS2)** implies that $\overline{\mathbf{C}}(z_1.z_2)$ is BIBO stable.

3.2 Stationary Points

As in the 1-D case [4] (see also [9]), we can establish that all locally stable stationary points of the given costs w.r.t. the combined composite system-equalizer impulse response $\{\overline{h}_j(k,l)\}$ are characterized by solutions such as (3-12) and (3-14) with the exception that now j_0 does not necessarily satisfy $|\overline{\gamma}_{rwj_0}| = |\overline{\gamma}_{rmax}|$. Moreover, if doubly-infinite 2-D equalizers are used then all locally stable stationary points of the given costs w.r.t. the equalizer coefficients are also characterized by solutions such as (3-12) and (3-14). This suggests an iterative solution where we iterate on inputs, one at a time; this is discussed next.

4 Iterative "source separation"

The preceding discussion suggests an iterative solution where we iterate on inputs one-by-one. Maximization of (3-8) w.r.t. the equalizer $\mathbf{C}(z_1,z_2)$ leads to the solution (3-14) under the sufficient conditions **(SC1)**-**(SC2)**. Implicit in the preceding discussion is the assumption that $\mathbf{C}(z_1,z_2)$ is allowed to be doubly-infinite. Given (3-14) we can estimate and remove the contribution of $w_{j_0}(m,n)$ from (2-1). Then we have a MIMO system with N outputs but $M-1$ inputs (instead of M inputs as in (2-1)-(2-2)). Repeat the process, i.e., maximize (3-8) w.r.t. a new equalizer to get a solution $e(m,n) = d'w_{j_0'}(m-m_0',n-n_0')$ where $j_0' \in (\{1,2,\cdots,M\}-\{j_0\})$. This leads to the following procedure:

1. Maximize (3-8) w.r.t. $\mathbf{C}(z_1,z_2)$ to obtain (3-14).
2. Cross-correlate $\{e(m,n)\}$ (of (3-14)) with the given data (2-1) and define an estimate of $f_{ij_0}(\tau_1,\tau_2)$ as

$$\widehat{f}_{ij_0}(\tau_1,\tau_2) \;:=\; \frac{E\{y_i(m,n)e(m-\tau_1,n-\tau_2)\}}{E\{e^2(m,n)\}}. \qquad (4-1)$$

   The reconstructed contribution $\widehat{y}_{i,j_0}(m,n)$ of $e(m,n)$ to the data $y_i(m,n)$ ($i = 1,2,\cdots,M$), is

$$\widehat{y}_{i,j_0}(m,n) := \sum_k\sum_l \widehat{f}_{ij_0}(k,l)e(m-k,n-l). \qquad (4-2)$$

3. Remove the above contribution from the data to define the outputs of a MIMO system with N outputs and $M-1$ inputs. These are given by

$$y_i'(m,n) \;:=\; y_i(m,n) - \widehat{y}_{i,j_0}(m,n). \qquad (4-3)$$

4. If $M>1$, set $M \leftarrow M-1$, $y_i(m,n) \leftarrow y_i'(m,n)$, and go back to Step **1**, else quit.

Analyzing the above algorithm we have

$$E\{y_i(m,n)e(m-\tau_1,n-\tau_2)\}$$

$$= \sum_{j=1}^{M} F_{ij}(z_1,z_2)E\{w_j(m,n)e(m-\tau_1,n-\tau_2)\}$$

$$= F_{ij_0}(z_1,z_2)d\sigma_w^2\delta(k_0+\tau_1,l_0+\tau_2)$$

$$= f_{ij_0}(k_0+\tau_1,l_0+\tau_2)d\sigma_w^2. \qquad (4-4)$$

Using (4-4) in (4-1) we have

$$\widehat{f}_{ij_0}(\tau_1,\tau_2) = \frac{f_{ij_0}(k_0+\tau_1,l_0+\tau_2)d\sigma_w^2}{d^2\sigma_w^2}$$

$$= f_{ij_0}(k_0+\tau_1,l_0+\tau_2)/d. \qquad (4-5)$$

It follows from (4-2) and (4-5) that

$$\widehat{y}_{i,j_0}(m,n) = \sum_k\sum_l f_{ij_0}(k,l)w_{j_0}(m-k,n-l). \qquad (4-6)$$

For $i = 1, 2, \cdots, N$, it follows from (4-3) and (4-6) that

$$y_i'(m,n) = \sum_{j=1,j\neq j_0}^{M} F_{ij}(z_1,z_2)w_j(m,n). \qquad (4-7)$$

5 Optimization

We will use an iterative, batch, steepest descent (ascent) method for maximization of (3-8). This cost is invariant to any scaling of the equalizer. In order to fix this, at every iteration we normalize equalizer taps to unit norm. Let $\mathbf{c}$ denote the vector of equalizer taps and let $\widehat{J}_{r2}(\mathbf{c})$ denote the data-based (3-8) with its explicit dependence upon $\mathbf{c}$.

(0) Let $\mathbf{c}$ denote the initial guess which we take to be "center" tap set to one and all the remaining taps set to zero.

(i) Set $\rho = 1$.

(ii) Calculate $\mathbf{c}' = \mathbf{c}+\rho\frac{\partial\widehat{J}_{r2}(\mathbf{c})}{\partial\mathbf{c}}$ and the resulting cost $\widehat{J}_{r2}(\mathbf{c}')$.

(iii) If $\widehat{J}_{r2}(\mathbf{c}') > \widehat{J}_{r2}(\mathbf{c})$, then accept $\mathbf{c}'' = \mathbf{c}'/\parallel\mathbf{c}'\parallel$ as the new equalizer tap vector, set $\mathbf{c} \leftarrow \mathbf{c}''$, and go to (i). Else set $\rho = \rho/2$ and go to (ii).

6 Identifiability

It follows from the preceding developments (Secs. 3 and 4) that under the conditions (**AS1**), (**AS2**) and (**SC1**)-(**SC2**), the proposed iterative approach is capable of blind identification of a 2-D MIMO transfer function $\mathbf{F}(z_1,z_2)$ (see (2-1)-(2-4)) up to a time-shift, a scaling and a permutation matrix provided that we allow doubly-infinite equalizers (in both dimensions). That is, given $\mathbf{F}(z_1,z_2)$, we end up with a $\mathbf{G}(z_1,z_2)$ where the two are related via

$$\mathbf{G}(z_1,z_2) = \mathbf{F}(z_1,z_2)\mathbf{DAP} \qquad (6-1)$$

where $\mathbf{D}$ is a $M\times M$ "spatial-shift" diagonal matrix (recall (k_0,l_0) in (3-12)), $\mathbf{A}$ is a $M\times M$ diagonal scaling matrix (recall d in (3-12)), and $\mathbf{P}$ is a $M\times M$ permutation matrix (recall j_0 in (3-12), we don't "know" which input it refers to). [A permutation matrix $\mathbf{P}$ has a single nonzero entry (equaling one) in each row and column, i.e., $P_{ij}=1$ if $j=j_i$, and $P_{ij}=0$ if $j\neq j_i$, $(i,j=1,2,\cdots,M)$, where P_{ij} is the ij-th element of $\mathbf{P}$.]

Theorem 1. Given the model (2-2) such that $\mathbf{v}(m,n)\equiv 0$ and given the true r-th order cumulant functions of the model output $\{\mathbf{y}(m,n)\}$ for r=2 and r=3 or 4, such that conditions (**AS1**), (**AS2**) and (**SC1**)-(**SC2**) hold true. Suppose that doubly-infinite equalizers are used in steps 1-4 of the iterative procedure of Sec. 4. Then this procedure yields a transfer function $\mathbf{G}(z_1,z_2)$ satisfying (6-1). •

Remark. The above discussion is couched in terms of the system impulse response implying an MA model fitting procedure. It need not be so. As in [2] other parametric models (AR or ARMA) can be fitted. Use of "long" MA equalizers to extract driving sequences $w_j(m,n)$ one-by-one has the practical advantage that all stable stationary points w.r.t. the equalizer coefficients correspond to the desired solution! □

7 Conclusions

In this paper we extended the approach of [2] (which deals with SISO 2-D systems) to MIMO 2-D systems. The paper was focused on certain theoretical aspects of the problem: estimation criteria, existence of a solution and parameter identifiability. Computational experience using the proposed approach has yet to be gained.

8 References

[1] A. Swami, G.B. Giannakis, and J.M. Mendel, "Linear modeling of multidimensional non-Gaussian processes using cumulants," *Multidimensional Systems and Signal Processing*, vol. 1, pp. 11-37, March 1990.

[2] J.K. Tugnait, "Estimation of linear parametric models of non-Gaussian discrete random fields with application to texture synthesis," *IEEE Trans. Image Processing*, vol. IP-3, pp. 109-127, March 1994.

[3] T.E. Hall and G.B. Giannakis, "Bispectral analysis and model validation of image textures," *IEEE Trans. Image Processing*, vol. IP-3, July 1995.

[4] J.K. Tugnait, "Identification of multichannel linear non-Gaussian processes using higher-order statistics," in *Proc. 29th Annual Asilomar Conf. Signals Systems Computers*, pp. 792-797, Pacific Grove, CA, Oct. 29 - Nov. 1, 1995.

[5] N.P. Galatsanos and R.T. Chin, "Digital restoration of multichannel images," *IEEE Trans. Signal Processing*, vol. SP-37, pp. 415-421, March 1989.

[6] A.K. Katsaggelos, K.T. Lay and N.P. Galatsanos, "A general framework for frequency domain multichannel signal processing," *IEEE Trans. Image Processing*, vol. IP-2, pp. 417-420, July 1993.

[7] A.K. Jain, *Fundamentals of Digital Image Processing*. Prentice-Hall: Englewood Cliffs, N.J., 1989.

[8] T. Kailath, *Linear Systems*. Englewood Cliffs, NJ: Prentice-Hall, 1980.

[9] J.K. Tugnait, "Identification and deconvolution of multichannel linear non-Gaussian processes using higher-order statistics and inverse filter criteria," *IEEE Trans. Signal Processing*, to appear in 1997.