CLOSED-FORM MULTI-DIMENSIONAL MULTI-INVARIANCE ESPRIT [†]

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ABSTRACT

A closed-form multi-dimensional multi-invariance generalization of the ESPRIT algorithm is introduced to exploit the *entire* invariance structure underlying a (possibly) multi-parametric data model, thereby greatly improving estimation performance. The multiple-invariance data structure that this proposed method can handle includes: (1) multiple occurrence of one size of invariance along one or multiple parametric dimensions, (2) multiple sizes of invariances along one or multiple parametric dimensions, and (3) invariances that cross over two or more parametric dimensions. The basic (uni-dimensional uni-invariance) ESPRIT algorithm is applied in parallel to each multiple pair of matrix-pencils characterizing the multiple invariance relationships in the data model, producing multiple sets of cyclically ambiguous estimates over the multi-dimensional parameter space. A weighted least-squares hyper-plane is then fitted to these set of estimates to yield very accurate and unambiguous estimates of the signal parameters.

1. INTRODUCTION

ESPRIT [1] (Estimation of Signal Parameters via Rotational Invariance Techniques) represents a highly popular eigenstructure (subspace) method. ESPRIT has found wide-ranging applications, from radar, to sonar, wireless cellular communications, global positioning systems (GPS), microwave imaging and even image analysis. However, ESPRIT, as originally formulated, exploits only one invariance per parametric dimension. Any multiple-invariance structure — including multiple occurrences of one size of invariance along the same parametric dimension, or multiple sizes of invariances along the same parametric dimension, or invariances that cross over two or more parametric dimensions, or any combination of these above cases — embedded in the data set would be ignored by ESPRIT in its original formulation. Such multi-dimensional multi-invariance data structure appears in many practical applications. Overlooking such data's full invariance structure compromises estimation performance.

For example, consider the two-dimensional direction-finding problem with a $L_x \times L_y$ rectangular array uniformly spaced at Δ. This is a two-dimensional problem estimating the direction-cosines along the x-axis and the y-axis, $\{u_k = \sin\theta_k \cos\phi_k, v_k = \sin\theta_k \sin\phi_k,\ k = 1, \ldots, K\}$ (where θ_k is the elevation angle measured from the z-axis, ϕ_k is the azimuth angle measured from the x-axis). Along each parametric dimension, there exist multiple sizes of spatial invariances, ranging from $\Delta^{(d)}$, $2\Delta^{(d)}$, and so on all the way to $(L-1)\Delta^{(d)}$, where $d = x$ or $d = y$. Moreover, each of these sizes of spatial invariances may be regarded as occurring multiple times. Moreover, there also exist various sizes of "diagonally oriented" spatial invariances at various angles with the x-y coordinates.

2. REVIEW OF RELEVANT LITERATURE

Open-form iterative search methods have been proposed to extend the *general* uni-invariance ESPRIT algorithm to the multi-parametric multi-invariance case, e.g. Roy, Ottersten, Swindlehurst & Kailath [2] for the uni-dimensional multiple-invariance case, Swindlehurst, Roy & Kailath [3] and Swindlehurst & Kailath [4,5] for the multi-dimensional multiple-invariance case. These iterative searches require complete knowledge (and thus computer storage) of the array manifold function's dependency on signal parameters. However, the foregoing methods do not fully exploit ESPRIT's main advantages of closed-form solution and freedom from having to store in computer memory detailed information of the array manifold function. Instead in the present algorithm, the basic (uni-dimensional uni-invariance) ESPRIT algorithm is applied in parallel to each multiple pair of matrix-pencils characterizing the multiple invariance relationships in the data model, producing multiple sets of cyclically ambiguous estimates over the multi-dimensional parameter space. A weighted least-squares hyper-plane is then fitted to these set of estimates to yield very accurate and unambiguous estimates of the signal parameters.

3. MATHEMATICAL DATA MODEL

Whilst the following presentation is developed from the perspective of sensor-array direction finding, the present algorithm is applicable to *any* multi-dimensional multi-invariance parameter estimation problem. The sampled data consist of a time sequence of $L \times 1$ measurements $\{\mathbf{z}(t_n),\ ,n = 1, \ldots, N\}$. At each time instance t_n:

$$\mathbf{z}(t_n) \stackrel{\text{def}}{=} \mathbf{a}(u_k^{(1)}, \ldots, u_k^{(D)}) \underbrace{\sqrt{\mathcal{P}_k}\sigma_k(t_n)e^{j(2\pi f t_n + \varphi_k)}}_{\stackrel{\text{def}}{=} s_k(t_n)} + \mathbf{n}(t_n)$$

where $\mathbf{a}(u_k^{(1)}, \ldots, u_k^{(D)})$ represents the $L \times 1$ array manifold in response to the k-th excitation source, which is parameterized by its D Cartesian direction-cosines $\{u_k^{(1)}, \ldots, u_k^{(D)}\}$, where $D = 1, 2,$ or 3. [1] Also, $\mathcal{P}_k$ denotes the k-th excitation source's power, $\sigma_k(t)$ is a zero-mean unit-variance complex random process, f refers to the sources' excitation frequency, φ_k denotes the k-th signal's uniformly-distributed random carrier phase, and $\mathbf{n}(t_n)$ is an $L \times 1$ noise vector for additive zero-mean white noise with variance σ_n^2.

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[1] A temporal invariance also exists if the incident signals are monochromatic pure tones.

For example, an $L_x \times L_y$-element rectangular array of identical omni-directional sensors uniformly spaced at Δ^x and Δ^y would have the array manifold:

$$\mathbf{a}(u_k, v_k) = \begin{bmatrix} 1 \\ e^{j2\pi\frac{\Delta^x}{\lambda}u_k} \\ \vdots \\ e^{j2\pi(L_x-1)\frac{\Delta^x}{\lambda}u_k} \end{bmatrix} \otimes \begin{bmatrix} 1 \\ e^{j2\pi\frac{\Delta^y}{\lambda}v_k} \\ \vdots \\ e^{j2\pi(L_y-1)\frac{\Delta^y}{\lambda}v_k} \end{bmatrix}$$

where $u_k = \sin\theta_k \cos\phi_k$ represents the k-th source's direction cosine relative to the x-axis, and $v_k = \sin\theta_k \sin\phi_k$ represents the k-th source's direction cosine relative to the y-axis, $0 \le \phi_k < 2\pi$ denotes the k-th source's azimuth angle, and $0 \le \theta_k < \pi/2$ denotes the k-th source's elevation angle.

Back to the general case, if $N > K$ snapshots are taken, then the entire data set $\mathbf{Z}$ is $L \times N$ in size and may be expressed as:

$$\begin{aligned} \mathbf{Z} &\stackrel{\text{def}}{=} [\mathbf{z}(t_1), \ldots, \mathbf{z}(t_N)] &(1)\\ &= \mathbf{A}(u_1^{(1)}, \ldots, u_1^{(D)}, \ldots\ldots, u_K^{(1)}, \ldots, u_K^{(D)})\mathbf{s}(t) + \mathbf{n}(t) \end{aligned}$$

where

$$\begin{aligned} &\mathbf{A}(u_1^{(1)}, \ldots, u_1^{(D)}, \ldots\ldots, u_K^{(1)}, \ldots, u_K^{(D)}) &(2)\\ &\stackrel{\text{def}}{=} \left[\mathbf{a}(u_1^{(1)}, \ldots, u_1^{(D)}), \cdots, \mathbf{a}(u_K^{(1)}, \ldots, u_k^{(D)})\right] \end{aligned}$$

$$\mathbf{s}(t) \stackrel{\text{def}}{=} \begin{bmatrix} s_1(t) \\ \vdots \\ s_K(t) \end{bmatrix}; \quad \mathbf{n}(t) \stackrel{\text{def}}{=} \begin{bmatrix} n_1(t) \\ \vdots \\ n_L(t) \end{bmatrix}$$

4. REVIEW OF UNI-DIMENSIONAL UNI-INVARIANCE ESPRIT

Assuming that the array manifold is unambiguous (i.e., a one-to-one mapping exists between u_k and $\mathbf{a}(u_k)$, or equivalently $\mathbf{a}(u_i) \neq \mathbf{a}(u_k)$ for all $i \neq k$), $\mathbf{Z}$ has rank equal to K under noiseless or asymptotic scenarios. That is, the column-space of $\mathbf{Z}$ may be decomposed into a K-dimensional signal-subspace and an $(L-K)$-dimensional noise-subspace. It follows that there exists a full-rank $L \times K$ signal-subspace eigenvector matrix $\mathbf{E}_s$ and a non-singular $K \times K$ coupling matrix $\mathbf{T}$ such that: $\mathbf{E}_s = \mathbf{A}(u_1, \ldots, u_K)\mathbf{T}$.

Suppose this L-element array has been so configured to contain two identical but translated (and possibly overlapping) subarrays. Two $\tilde{L} \times 1$ subarray manifolds $\mathbf{a}^{(1)}(u_k)$ and $\mathbf{a}^{(2)}(u_k)$ may be formed out of the $L \times 1$ array manifold $\mathbf{a}(u_k)$ according to:

$$\mathbf{a}^{(1)}(u_k) \stackrel{\text{def}}{=} \mathbf{J}^{(1)}\mathbf{a}(u_k) \quad (3)$$

$$\mathbf{a}^{(2)}(u_k) \stackrel{\text{def}}{=} \mathbf{J}^{(2)}\mathbf{a}(u_k) \quad (4)$$

where $\mathbf{J}^{(1)}$ and $\mathbf{J}^{(2)}$ are $\tilde{L} \times L$ full-rank subarray selection matrices. Because the two subarrays have been required to be identical but translated by Δ, the two subarray manifolds are related by the invariance phase factor, $q(u_k) \stackrel{\text{def}}{=} e^{j2\pi\frac{\Delta}{\lambda}u_k}$:

$$\mathbf{a}^{(2)}(u_k) = \mathbf{a}^{(1)}(u_k)q(u_k) \quad (5)$$

where λ is the excitation sources' wavelength.

Two $\tilde{L} \times K$ signal-subspace eigenvector sub-matrices $\mathbf{E}_s^{(1)}$ and $\mathbf{E}_s^{(2)}$ may be formed using the same subarray selection matrices $\mathbf{J}^{(1)}$ and $\mathbf{J}^{(2)}$:

$$\mathbf{E}_s^{(1)} \stackrel{\text{def}}{=} \mathbf{J}^{(1)}\mathbf{E}_s = \underbrace{\mathbf{J}^{(1)}\mathbf{A}}_{\stackrel{\text{def}}{=}\mathbf{A}^{(1)}}\mathbf{T} \quad (6)$$

$$\mathbf{E}_s^{(2)} \stackrel{\text{def}}{=} \mathbf{J}^{(2)}\mathbf{E}_s = \underbrace{\mathbf{J}^{(2)}\mathbf{A}}_{\stackrel{\text{def}}{=}\mathbf{A}^{(2)}}\mathbf{T} \quad (7)$$

This matrix-pencil pair $\{\mathbf{E}_s^{(1)}, \mathbf{E}_s^{(2)}\}$ is related by a $K \times K$ non-singular matrix $\mathbf{\Psi}$:

$$\mathbf{E}_s^{(1)} = \mathbf{E}_s^{(2)}\mathbf{\Psi} \quad (8)$$

$\mathbf{\Psi}$ must be non-singular because both $\mathbf{E}_s^{(1)}$ and $\mathbf{E}_s^{(2)}$ are full-rank. However, it is also true that

$$\mathbf{A}^{(1)} = \mathbf{A}^{(2)}\mathbf{\Phi}, \quad \mathbf{\Phi} \stackrel{\text{def}}{=} \begin{bmatrix} e^{j2\pi\frac{\Delta}{\lambda}u_1} & & \\ & \ddots & \\ & & e^{j2\pi\frac{\Delta}{\lambda}u_K} \end{bmatrix}$$

$$\begin{aligned} \Rightarrow \quad \mathbf{E}_s^{(1)}\mathbf{T}^{-1} &= \mathbf{E}_s^{(2)}\mathbf{T}^{-1}\mathbf{\Phi} \\ \Rightarrow \quad \mathbf{\Psi} &= \left[\left(\mathbf{E}_s^{(1)H}\mathbf{E}_s^{(1)}\right)^{-1}\mathbf{E}_s^{(1)H}\right]\mathbf{E}_s^{(2)} = \mathbf{T}^{-1}\mathbf{\Phi}\mathbf{T} \end{aligned}$$

That is, the k-th diagonal element $[\mathbf{\Phi}]_{kk}$ of the $K \times K$ diagonal matrix $\mathbf{\Phi}$ equals the k-th eigenvalue of $\mathbf{\Psi}$, and its corresponding eigenvector constitutes the k-th column of $\mathbf{T}^{-1}$. In noiseless or asymptotic cases, all the above relations are exact. This means that when $\Delta \le \frac{\lambda}{2}$, the sources' parameters $\{u_k, \quad k = 1, \ldots, K\}$ may be estimated as:

$$\hat{u}_k \stackrel{\text{def}}{=} \frac{\angle[\mathbf{\Phi}]_{kk}}{2\pi\frac{\Delta}{\lambda}} \qquad k = 1, \ldots, K \quad (9)$$

where $\angle[\mathbf{\Phi}]_{kk}$ denotes the principal argument of the phase of $[\mathbf{\Phi}]_{kk}$ between $-\pi$ and π.

Note also that ESPRIT does not directly estimate u_k, but only the invariant phase factors $e^{j2\pi\frac{\Delta}{\lambda}u_k}$. That is, when $\Delta > \frac{\lambda}{2}$, ESPRIT only yields cyclically ambiguous estimates of u_k (i.e., $\mu_k \stackrel{\text{def}}{=} [(\frac{\Delta}{\lambda}u_k + 1) \bmod 2] - 1)$ but not u_k directly. This means that for $\Delta > \frac{\lambda}{2}$, no one-to-one mapping exists between u_k and μ_k. What exists for each Δ is a set of cyclically related candidates for u_k:

$$\check{u}_k(n_k) = \mu_k + n_k\frac{\lambda}{\Delta} \quad (10)$$

where n_k is an integer in the range

$$\lceil(-1-\mu_k)\frac{\Delta}{\lambda}\rceil \le n_k \le \lfloor(1-\mu_k)\frac{\Delta}{\lambda}\rfloor \quad (11)$$

Note that this embodies a uni-dimensional problem because only one unknown parameter (u_k) needs to be estimated for the k-th source. This is also a uni-invariance problem because $e^{j2\pi\frac{\Delta}{\lambda}u_k}$ represents the only invariance factor in the problem.

5. THE MULTI-DIMENSIONAL MULTIPLE-SIZE INVARIANCE PROBLEM

Suppose there exist I distinct pairs of subarray manifolds, $\{\mathbf{a}_i^{(1)}(u_k^{(1)}, \ldots, u_k^{(D)}), \mathbf{a}_i^{(2)}(u_k^{(1)}, \ldots, u_k^{(D)}),\ i = 1, \ldots, I\}$, of size $\tilde{L}_i \times 1$ respectively formed using the subarray selection matrices $\{\mathbf{J}_i^{(1)}, \mathbf{J}_i^{(2)},\ i = 1, \ldots, I\}$:

$$\begin{aligned} \mathbf{a}_i^{(1)}(u_k^{(1)}, \ldots, u_k^{(D)}) &\stackrel{\text{def}}{=} \mathbf{J}_i^{(1)}\mathbf{a}(u_k^{(1)}, \ldots, u_k^{(D)}) \\ \mathbf{a}_i^{(2)}(u_k^{(1)}, \ldots, u_k^{(D)}) &\stackrel{\text{def}}{=} \mathbf{J}_i^{(2)}\mathbf{a}(u_k^{(1)}, \ldots, u_k^{(D)}) &(12)\\ &= \mathbf{a}_i^{(1)}(u_k^{(1)}, \ldots, u_k^{(D)})q_i(u_k^{(1)}, \ldots, u_k^{(D)}) \\ q_i(u_k^{(1)}, \ldots, u_k^{(D)}) &\stackrel{\text{def}}{=} e^{j2\pi[\Sigma_{d=1}^{D}\frac{\Delta_i^{(d)}}{\lambda}u_k^{(d)}]} \end{aligned}$$

where D is the total number of parametric dimensions, $u_k^{(d)}$ is the d-th of the D parameters characterizing the k-th impinging source. This represents a D-dimensional invariance problem because the k-th source is parameterized by D distinct parameters, namely $\{u_k^{(1)}, \ldots, u_k^{(D)}\}$.

This represents a multi-size invariance problem because the invariances $\{\Delta_i^{(d)}, \quad, d = 1, \ldots, D;\ i = 1, \ldots, I\}$ are not all equal.

This case permits the exploitation of all spatial invariances regardless of their sizes or their angular orientation with respect to the coordinate axes. The i-th matrix-pencil is obtained from two identical subarrays with the displacement vector between them having a magnitude equal to

$$\Delta_i \stackrel{\text{def}}{=} \sqrt{(\Delta_i^{(1)})^2 + \ldots + (\Delta_i^{(D)})^2} \tag{13}$$

oriented along the direction $(\Delta_i^{(1)}, \ldots, \Delta_i^{(D)})$. The corresponding direction-cosine is:

$$w_{i,k} \stackrel{\text{def}}{=} \sum_{d=1}^{D} \frac{\Delta_i^{(d)} u_k^{(d)}}{\Delta_i} \tag{14}$$

Application of ESPRIT to this i-th matrix-pencil pair yields $\omega_{i,k} \stackrel{\text{def}}{=} \left[(\sum_{d=1}^{D} \Delta_i^{(d)} u_k^{(d)} + 1) \mod 2 \right] - 1$ as estimate of $w_{i,k}$, but not of $\{u_k^{(1)}, \ldots, u_k^{(D)}, k, = 1, \ldots, K\}$ directly. This estimate for the direction-cosine $w_{i,k}$ along the direction $(\Delta_i^{(1)}, \ldots, \Delta_i^{(D)})$ would be cyclically ambiguous if $\Delta_i > \frac{\lambda}{2}$ due to the spatial Nyquist Sampling Theorem. That is, along this $(\Delta_i^{(1)}, \ldots, \Delta_i^{(D)})$ direction, there exists for $w_{i,k}$ the cyclically related set of candidate estimates:

$$\check{w}_{i,k}(n_{i,k}) = \omega_{i,k} + n_{i,k} \frac{\lambda}{\Delta_i} \tag{15}$$

for an unknown $n_{i,k}$ in the range

$$\lceil \frac{\Delta_i}{\lambda}(-1 - \omega_{i,k}) \rceil \leq n_{i,k} \leq \lfloor \frac{\Delta_i}{\lambda}(1 - \omega_{i,k}) \rfloor \tag{16}$$

Relating to the azimuth-elevation direction-finding problem with a uniform rectangular array, modify the notation from $\{u_k^{(1)}, u_k^{(2)}\}$ to $\{u_k, v_k\}$ and from $\{\Delta_i^{(1)}, \Delta_i^{(2)}\}$ to $\{\Delta_i^x, \Delta_i^y\}$. The (i,j)-th pair of subarray selection matrices $\mathbf{J}_i^{(1)}$ and $\mathbf{J}_i^{(2)}$:

$$\mathbf{J}_i^{(1)} \stackrel{\text{def}}{=} \left[\mathbf{I}_{i_x} \mid \mathbf{O}_{(i_x, L_x - i_x)} \right] \otimes \left[\mathbf{I}_{i_y} \mid \mathbf{O}_{(i_y, L_y - i_y)} \right] \tag{17}$$

$$\mathbf{J}_i^{(2)} \stackrel{\text{def}}{=} \left[\mathbf{O}_{(i_x, L_x - i_x)} \mid \mathbf{I}_{i_x} \right] \otimes \left[\mathbf{O}_{(i_y, L_x - i_y)} \mid \mathbf{I}_{i_y} \right], \tag{18}$$

$$\text{where} \quad 1 \leq i_x < L_x - K, \quad 1 \leq i_y < L_y - K$$

may be used to form the subarray manifolds:

$$\mathbf{a}_i^{(1)}(u_k, v_k) \stackrel{\text{def}}{=} \mathbf{J}_i^{(1)} \mathbf{a}(u_k, v_k) \tag{19}$$

$$\mathbf{a}_i^{(2)}(u_k, v_k) \stackrel{\text{def}}{=} \mathbf{J}_i^{(2)} \mathbf{a}(u_k, v_k) \tag{20}$$

linked by the invariant phase factor $q_i(u_k, v_k) = e^{j2\pi(\frac{\Delta_i^x}{\lambda} u_k + \frac{\Delta_i^y}{\lambda} v_k)}$

$$\mathbf{a}_i^{(2)}(u_k, v_k) = \underbrace{e^{j2\pi(\frac{\Delta_i^x}{\lambda} u_k + \frac{\Delta_i^y}{\lambda} v_k)}}_{\stackrel{\text{def}}{=} q_i(u_k, v_k)} \mathbf{a}_i^{(1)}(u_k, v_k),$$

$$1 \leq i \leq I \leq L_x - K$$

This (i,j)-th invariance $\Delta_i \stackrel{\text{def}}{=} \sqrt{(\Delta_i^x)^2 + (\Delta_i^y)^2}$, is oriented along the direction $\{\Delta_i^x, \Delta_i^y\}$ and the corresponding direction cosine is:

$$w_{i,k} \stackrel{\text{def}}{=} \frac{\frac{\Delta_i^x}{\lambda} u_k + \frac{\Delta_i^y}{\lambda} v_k}{\Delta_i / \lambda} \tag{21}$$

If $\Delta_i > \frac{\lambda}{2}$, then the set of cyclically ambiguous estimates of $w_{i,k}$ is:

$$\check{w}_{i,k}(n_{i,k}) = \omega_{i,k} + n_{i,k} \frac{\lambda}{\Delta_i}, \tag{22}$$

$$\text{for} \quad \lceil (-1 - \omega_{i,k}) \frac{\Delta_i}{\lambda} \rceil \leq n_{i,k} \leq \lfloor (1 - \omega_{i,k}) \frac{\Delta_i}{\lambda} \rfloor \tag{23}$$

6. MULTI-DIMENSIONAL MULTI-INVARIANCE ESPRIT

Under noiseless or asymptotic conditions, the i-th matrix-pencil's eigenvalue for the k-th impinging source (i.e. $e^{j2\pi \frac{\Delta_i}{\lambda} w_{i,k}}$), when correctly disambiguated, corresponds to the phase

$$2\pi \frac{\Delta_i}{\lambda} \check{w}_{i,k}(n_{i,k}^{\circ}) \stackrel{\text{def}}{=} 2\pi \frac{\Delta_i}{\lambda} \omega_{i,k} + 2\pi n_{i,k}^{\circ} \tag{24}$$

which is an affine function of Δ_i. That is, for the k-th source, each of the I matrix-pencil pairs corresponds to one point $(\Delta_i^{(1)}, \ldots, \Delta_i^{(D)}; 2\pi \frac{\Delta_i}{\lambda} w_{i,k})$ on the k-th D-dimensional hyperplane defined as:

$$0 = \sum_{d=1}^{D} \frac{x^{(d)}}{u_k^{(d)}}, \qquad k = 1, \ldots, K \tag{25}$$

That is, all I diversely oriented matrix-pencil pairs together define for each incident source a separate D-dimensional hyperplane through the origin The k-th hyperplane passes through the origin and has the inverses of the Cartesian direction-cosines $\{u_k^{(1)}, \ldots, u_k^{(D)}\}$ as its coefficients in the Cartesian coordinate system $\{x^{(1)}, \ldots, x^{(D)}\}$. However, (25) is expressed as a function of the unambiguous Cartesian direction-cosine $u_k^{(d)}$, which are of course not available. Instead, available are only the cyclically ambiguous *non*-Cartesian direction-cosine estimates

$$\check{w}_{i,k}(n_{i,k}) = \omega_{i,k} + n_{i,k} \frac{\lambda}{\Delta_i}$$

$$\text{where} \quad \lceil (-1 - \omega_{i,k}) \frac{\Delta_i}{\lambda} \rceil \leq n_{i,k} \leq \lfloor (1 - \omega_{i,k}) \frac{\Delta_i}{\lambda} \rfloor, \quad 1 \leq i \leq I$$

The k-th hyperplane may be estimated as follows:

(1) Derive one hyperplane candidate that best fits (in the least squares sense) the set of points $\{(\Delta_i^{(1)}, \ldots, \Delta_i^{(D)}; \frac{\Delta_i}{\lambda} \check{w}_{i,k}(n_{i,k})),\ \text{for } i = 1, \ldots, I\}$.

For example, in the uniform rectangular array two-dimensional direction finding problem:

$$\tilde{u}_k(n_{1,k}, \ldots, n_{I,k}) = \frac{b_2 b_5 - b_3 b_4}{(b_2)^2 - b_1 b_4} \tag{26}$$

$$\tilde{v}_k(n_{1,k}, \ldots, n_{I,k}) = \frac{b_2 b_3 - b_1 b_5}{(b_2)^2 - b_1 b_4} \tag{27}$$

$$b_1 \stackrel{\text{def}}{=} \sum_{i=1}^{I} \left(\frac{\Delta_i^x}{\lambda} \right)^2 c_i, \qquad b_2 \stackrel{\text{def}}{=} \sum_{i=1}^{I} \frac{\Delta_i^x}{\lambda} \frac{\Delta_i^y}{\lambda} c_i \tag{28}$$

$$b_3 \stackrel{\text{def}}{=} \sum_{i=1}^{I} \frac{\Delta_i^x}{\lambda} \frac{\Delta_i}{\lambda} c_i \tilde{u}_k(n_{1,k}, \ldots, n_{I,k}) \tag{29}$$

$$b_4 \stackrel{\text{def}}{=} \sum_{i=1}^{I} \left(\frac{\Delta_i^y}{\lambda} \right)^2 c_i \tag{30}$$

$$b_5 \stackrel{\text{def}}{=} \sum_{i=1}^{I} \frac{\Delta_i^y}{\lambda} \frac{\Delta_i}{\lambda} c_i \tilde{v}(n_{1,k}, \ldots, n_{I,k}) \tag{31}$$

where c_i denotes the pre-determined weight assigned to the estimate $\omega_{i,k}$ obtained from the i-th matrix-pencil, and $\sum_{i=1}^{I} c_i = 1$. To minimize the estimation variance of $\{\hat{u}_k^{(1)}, \ldots, \hat{u}_k^{(D)}\}$, the weight c_i should be chosen to be inversely proportional to the expected variance of $\hat{\omega}_{i,k}$, thereby realizing a maximum-ratio combiner. The variances of $\{\omega_{i,k}, \, , i = 1, \ldots, I\}$, being functions of array geometry and the spatial invariances, may be readily pre-calculated off-line for certain "typical" scenarios prior to real-time data measurement.

(2) Then compute the least-squares fitting error for each D-dimensional hyperplane candidate in the above step.

For the uniform rectangular array two-dimensional direction finding problem: $\mathrm{MSE}_k(n_{1,k}, \ldots, n_{I,k}) \stackrel{\text{def}}{=}$

$$\left\| \sum_{i=1}^{I} c_i \left(\frac{\Delta_i}{\lambda} \tilde{w}_{i,k}(n_{i,k}) - \frac{\Delta_i^x}{\lambda} \tilde{u}_k(n_{1,k}, \ldots, n_{I,k}) - \frac{\Delta_i^y}{\lambda} \tilde{v}_k(n_{1,k}), \ldots, n_{I,k}) \right)^2 \right\|$$

for $\lceil (-1 - \omega_{i,k}) \frac{\Delta_i}{\lambda} \rceil \leq n_{i,k} \leq \lfloor (1 - \omega_{i,k}) \frac{\Delta_i}{\lambda} \rfloor ; \; 1 \leq i \leq I$

(3) Identify the one hyperplane candidate above with the minimum fitting error as the estimate for the k-th hyperplane. Note that in noiseless or asymptotic cases, only the true hyperplane would have zero fitting error if at least D of the I matrix-pencils have linearly independent Cartesian invariances, i.e. if

$$\begin{bmatrix} \Delta_1^{(1)} & \cdots & \Delta_I^{(1)} \\ \vdots & & \vdots \\ \Delta_1^{(D)} & \cdots & \Delta_I^{(D)} \end{bmatrix}$$

is full rank.

If $\frac{\Delta_i}{\lambda} \gg 1$ for any $\{i = 1, \ldots, I\}$, then the set $\{n_{1,k}, \ldots, n_{I,k}\}$ may take on a very large possible number of values. The journal version of this work will present a short cut such that N_{i° instead of $\prod_{i=1}^{I} N_i$ candidates need to be considered, where i° denotes the invariance relation that would yield the most accurate estimate based on the aforementioned off-line calculations of the invariances of $\{\mu_{i,k}, \; i = 1, \ldots, I\}$.

The above procedures have assumed that the correct set of I direction-cosine estimates have been grouped for each of the K impinging sources. This pairing may be achievable, for example, by a procedure to be presented in the journal version of this work.

7. SIMULATIONS

Simulations presented in Figure 1 demonstrate the efficacy and performance of the proposed algorithm. Two closely-spaced equal-power narrowband uncorrelated sources from $\{u_1 = 0.73, v_1 = 0.57\}$ and $\{u_2 = 0.79, v_2 = 0.51\}$ impinge upon a 15×15 rectangular array uniformly spaced at half-wavelength. The closed-form multi-invariance algorithm computes the twelve spatial invariances equal to $\{1\frac{\Delta^x}{\lambda}, 0\frac{\Delta^y}{\lambda}\}$, $\{0\frac{\Delta^x}{\lambda}, 1\frac{\Delta^y}{\lambda}\}$, $\{2\frac{\Delta^x}{\lambda}, 0\frac{\Delta^y}{\lambda}\}$, $\{0\frac{\Delta^x}{\lambda}, 2\frac{\Delta^y}{\lambda}\}$, $\{1\frac{\Delta^x}{\lambda}, 1\frac{\Delta^y}{\lambda}\}$, $\{2\frac{\Delta^x}{\lambda}, 2\frac{\Delta^y}{\lambda}\}$, $\{1\frac{\Delta^x}{\lambda}, 2\frac{\Delta^y}{\lambda}\}$, $\{2\frac{\Delta^x}{\lambda}, 1\frac{\Delta^y}{\lambda}\}$, $\{3\frac{\Delta^x}{\lambda}, 0\frac{\Delta^y}{\lambda}\}$, $\{0\frac{\Delta^x}{\lambda}, 3\frac{\Delta^y}{\lambda}\}$, $\{1\frac{\Delta^x}{\lambda}, 3\frac{\Delta^y}{\lambda}\}$, and $\{3\frac{\Delta^x}{\lambda}, 1\frac{\Delta^y}{\lambda}\}$. The least-squares weights used for these twelve spatial invariances are respectively 0.1494, 0.1377, 0.0994, 0.0883, 0.0801, 0.0537, 0.0710, 0.0716, 0.0722, 0.0639, 0.0540 and 0.0587. The 12 sets of direction cosine estimates obtained from the 12 matrix-pencils have been assumed to be correctly paired. There are 100 snapshots per experiment and 200 independent experiments per data point. The RMS standard deviation plotted is equal to the square root of the mean samples variances for $\{\hat{u}_k, \hat{v}_k, \; k = 1, 2\}$. A 30% to 60% reduction in estimation standard deviation is produced by the proposed closed-form multi-invariance method relative to that of the single-invariance (per parametric dimension) method. The price paid is a fairly dramatic increase in computational complexity.

8. BIBLIOGRAPHY

[1] R. Roy and T. Kailath, "ESPRIT-Estimation of Signal Parameters Via Rotational Invariance Techniques," *IEEE Trans. Acoustics, Speech, Signal Proc.*, vol. 37, no. 7, pp. 984-995, 7/89.

[2] R. Roy, B. Ottersten, A. L. Swindlehurst and T. Kailath, "Multiple Invariance ESPRIT," *22nd Asilomar Conf.*, pp. 583-587, 1988. *IEEE Trans. Signal Proc.*, vol. 40, no. 4, pp. 867-881, 4/92.

[3] A. L. Swindlehurst, R. Roy and T. Kailath, "Suboptimal Subspace Fitting Methods for Multidimensional Signal Parameter Estimation," *Proc. SPIE Intl. Soc. Opt. Eng.*, vol 1152, pp. 197-208, 1989.

[4] A. L. Swindlehurst and T. Kailath, "2-D Parameter Estimation Using Arrays with Multidimensional Invariance Structure," *23rd Asilomar Conf.*, pp. 950-954, 1989.

[5] A. L. Swindlehurst and T. Kailath, "Azimuth/Elevation Direction-Finding Using Regular Array Geometry," *IEEE Trans. Aerospace & Electronic Systems*, vol. 29, no. 1, pp. 145-156, 1/93.

Figure 1: Proposed algorithm's superior performance.