

ARRAY MANIFOLD MEASUREMENT IN THE PRESENCE OF MULTIPATH

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ABSTRACT

We present an algorithm for the calibration of sensor arrays in the presence of multipath. The algorithm is based on two sets of calibration data obtained from two angularly separated transmitting points. Simulation results demonstrating the performance of the algorithm are included.

1. INTRODUCTION

Modern superresolution direction finding techniques such as MUSIC [1], Maximum Likelihood [7], and subspace fitting methods [5], presume the knowledge of the array response.

As the analysis of these techniques show, [3], [4], any inaccuracy in the presumed array response results in severe degradation of performance. The measurement of the array response, referred to as array calibration, is therefore a crucial step in the implementation of these techniques.

The existing calibration techniques, [2], [6], are based on modeling the array response by a free-space model perturbed by an unknown coupling matrix and sensor location uncertainty. In this paper we address the problem of measuring the array response in the presence of multipath. This problem is important since multipath is essentially unavoidable and it sets the limit on the achievable calibration accuracy.

2. PROBLEM FORMULATION

Let $\mathbf{a}(\theta)$ denote the $p \times 1$ vector of the array response (steering vector) to a source impinging from direction θ . The array calibration problem amounts to measuring $\mathbf{a}(\theta)$ for $\theta \in (0, 2\pi]$. It is usually performed by transmitting a signal from some location, rotating the array and measuring the array response at each angle. Unfortunately, in many cases the measured response is composed not only of the direct path from the transmitting

point to the array, but also, of multiple reflections from the surroundings, see figure 1.

To cope with the multipath problem we propose to carry out the calibration measurements from two different transmitting points. Assuming that the calibration consists of N measurements taken uniformly on $\theta \in (0, 2\pi]$ the measured $p \times 1$ vector at the angle $\theta_k = \frac{2\pi k}{N}$ $1 \leq k \leq N$, for the l 'th transmitting point ($l = 1, 2$) can be expressed as :

$$\mathbf{y}_l(\theta_k) = \sum_{i=1}^{r_l} \rho_{i,l} \mathbf{a}(\theta_k - \theta_{i,l}) + \mathbf{n}_l(\theta_k) \quad (1)$$

where

$\theta_{i,l}$ - the direction of the i 'th reflection, at the l 'th set, measured with respect to the direction of the calibrating source.

$\rho_{i,l}$ - the complex coefficient representing the phase shift and the amplitude of the i 'th reflection, at the l 'th set.

r_l - the number of reflections at the l 'th set.

$\mathbf{n}_l(\frac{2\pi k}{N})$ - the noise vector at the l 'th set.

Since the array manifold is measured relative to some arbitrary point we can assume without loss of generality that $\rho_{1,1} = 1$ and $\theta_{1,1} = \theta_{1,2} = 0^\circ$. Also since the reflecting objects remain fixed while the

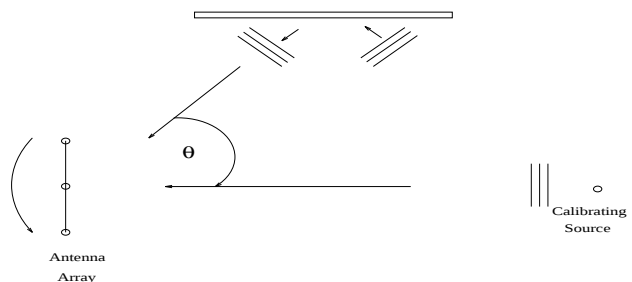


Figure 1. The Calibrating Signal with one Reflection.

transmitting point change, the relative directions of the reflections will change as well and hence $\theta_{i,1} \neq \theta_{i,2}$ ($i \neq 0$). The array calibration problem can now be formulated as follows. Given two measured data sets $\left\{ \mathbf{y}_l \left(\frac{2\pi k}{N} \right) \right\}_{k=1}^N$; $l = 1, 2$ estimate the array manifold $\left\{ \mathbf{a} \left(\frac{2\pi k}{N} \right) \right\}_{k=1}^N$.

3. THE PROPOSED SOLUTION

The proposed solution is based on the following steps :

- (i) Estimating the reflections' parameters $\left\{ \boldsymbol{\rho}_{i,l}, \theta_{i,l} \right\}$
- (ii) "Cleaning" the data by subtracting the reflections.

To solve the estimation problem, let $\mathbf{w}_l$ denote the $N \times 1$ vector whose k 'th element is given by $w_l(k) = \sum_{i=1}^{r_l} \rho_{i,l} \delta \left(\frac{2\pi k}{N} - \theta_{i,l} \right)$ where $\delta(\theta)$ is Dirac's delta function, and let $\mathbf{A}_l$ denote the $N \times N$ circulant matrix defined by $\mathbf{w}_l$

$$\mathbf{A}_l = \begin{bmatrix} w_l(1) & \cdots w_l(N-1) & w_l(N) \\ w_l(2) & \cdots w_l(N) & w_l(1) \\ \vdots & & \vdots \\ w_l(N) & w_l(1) \cdots & w_l(N-1) \end{bmatrix} \quad (2)$$

With this notation we can rewrite (1), for each sensor $1 \leq m \leq p$, in matrix form as

$$\mathbf{y}_{m,l} = \mathbf{A}_l \mathbf{a}_m + \mathbf{n}_{m,l} \quad (3)$$

where $\mathbf{a}_m$ is the $N \times 1$ array manifold of the m 'th sensor, $\mathbf{a}_m = [a_m(\frac{2\pi}{N}), \dots, a_m(\frac{2\pi N}{N})]^T$. Since $\mathbf{A}_l$ is a circulant matrix it is diagonalized by the DFT matrix of order N , and its eigenvalues are given by the DFT of the generating vector $\mathbf{w}_l$. Therefore

$$\mathbf{F}^H \mathbf{A}_l \mathbf{F} = \text{diag}\{\mathbf{F} \mathbf{w}_l\} = \text{diag}\{\hat{\mathbf{w}}_l\}, \quad (4)$$

where $\mathbf{F}$ is the DFT matrix of order N , and $\hat{\mathbf{w}}_l$ is the DFT of $\mathbf{w}_l$ given by

$$\hat{w}_l(k) = \sum_{i=1}^{r_l} \rho_{i,l} \omega_N^{k\theta_{i,l}} \quad (5)$$

with ω_N being the N 'th order primitive root of unity $\omega_N = e^{\frac{2\pi j}{N}}$. Hence

$$\mathbf{A}_l = \mathbf{F} \text{diag}\{\hat{\mathbf{w}}_l\} \mathbf{F}^H \quad (6)$$

Substituting this result into (3) and denoting

$$\mathbf{D}_l = \text{diag}\{\hat{\mathbf{w}}_l\} \quad (7)$$

we obtain

$$\mathbf{F} \mathbf{a}_m = \mathbf{D}_l^{-1} (\hat{\mathbf{y}}_{m,l} - \mathbf{n}_{m,l}) \quad (8)$$

where $\hat{\mathbf{y}}_{m,l}$ denotes the DFT of $\mathbf{y}_{m,l}$. Since this holds for both sets of measurements we obtain

$$\mathbf{D}_1^{-1} (\hat{\mathbf{y}}_{m,1} - \mathbf{n}_{m,1}) = \mathbf{D}_2^{-1} (\hat{\mathbf{y}}_{m,2} - \mathbf{n}_{m,2}) \quad (9)$$

which can be rewritten as

$$\hat{\mathbf{w}}_1 \circ \hat{\mathbf{y}}_{m,2} - \hat{\mathbf{w}}_2 \circ \hat{\mathbf{y}}_{m,1} = \hat{\mathbf{w}}_2 \circ \mathbf{n}_{m,1} - \hat{\mathbf{w}}_1 \circ \mathbf{n}_{m,2} \quad (10)$$

where $\circ$ denotes elementwise multiplication.

Based on this relation, a LS estimator for the directions of the reflections is given by

$$[\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\rho}}] = \min_{\boldsymbol{\theta}^+, \boldsymbol{\rho}^+} \|\hat{\mathbf{w}}_1 \circ \hat{\mathbf{y}}_{m,2} - \hat{\mathbf{w}}_2 \circ \hat{\mathbf{y}}_{m,1}\|^2 \quad (11)$$

where

$$\boldsymbol{\theta}^+ = [\theta_{1,1}, \dots, \theta_{r_1,1}, \theta_{1,2}, \dots, \theta_{r_2,2}]^T \quad (12)$$

$$\boldsymbol{\theta} = [\theta_{2,1}, \dots, \theta_{r_1,1}, \theta_{1,2}, \dots, \theta_{r_2,2}]^T \quad (13)$$

$$\boldsymbol{\rho}^+ = [\rho_{1,1}, \dots, \rho_{r_1,1}, \rho_{1,2}, \dots, \rho_{r_2,2}]^T \quad (14)$$

$$\boldsymbol{\rho} = [\rho_{2,1}, \dots, \rho_{r_1,1}, \rho_{1,2}, \dots, \rho_{r_2,2}]^T \quad (15)$$

and recalling that we assume

$$\rho_{1,1} = 1 \quad (16)$$

and

$$\theta_{1,1} = 0 \quad (17)$$

Substituting (5) into (11) yields

$$[\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\rho}}] = \min_{\boldsymbol{\theta}^+, \boldsymbol{\rho}^+} \left\| \hat{\mathbf{y}}_{m,2}(k) \sum_{i=1}^{r_1} \rho_{i,1} \omega_N^{k\theta_{i,1}} - \hat{\mathbf{y}}_{m,1}(k) \sum_{i=1}^{r_2} \rho_{i,2} \omega_N^{k\theta_{i,2}} \right\|^2 \quad (18)$$

Denoting

$$\Upsilon_l^m(k, \theta) = \hat{\mathbf{y}}_{m,l} \left(\frac{2\pi k}{N} \right) \omega_N^{k\theta}, \quad l = 1, 2$$

and

$$\Upsilon_l^m(\theta) = [\Upsilon_l^m(0, \theta), \dots, \Upsilon_l^m(N-1, \theta)]$$

we define

$$\mathbf{B}_m(\boldsymbol{\theta}) = \begin{bmatrix} \Upsilon_1^m(\theta_{2,1}) \\ \vdots \\ \Upsilon_1^m(\theta_{r_1,1}) \\ -\Upsilon_2^m(\theta_{1,2}) \\ \vdots \\ -\Upsilon_2^m(\theta_{r_2,2}) \end{bmatrix}^T$$

we can rewrite (18) as a linear problem in $\boldsymbol{\rho}$

$$[\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\rho}}] = \min_{\boldsymbol{\theta}, \boldsymbol{\rho}} \|\mathbf{B}_m(\boldsymbol{\theta})\boldsymbol{\rho} - \hat{\mathbf{y}}_{m,1}\|^2 \quad (19)$$

This estimator is based on the data of the m 'th sensor only. Clearly, we can improve the performance by combining the information from all sensors. This yields

$$[\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\rho}}] = \arg \min_{\boldsymbol{\theta}, \boldsymbol{\rho}} \|\mathbf{B}(\boldsymbol{\theta})\boldsymbol{\rho} - \hat{\mathbf{y}}_1\|^2 \quad (20)$$

where

$$\mathbf{B}(\boldsymbol{\theta}) = [\mathbf{B}_1(\boldsymbol{\theta})^T, \dots, \mathbf{B}_m(\boldsymbol{\theta})^T]^T \quad (21)$$

and

$$\hat{\mathbf{y}}_1 = [\hat{\mathbf{y}}_{1,1}^T, \dots, \hat{\mathbf{y}}_{p,1}^T]^T \quad (22)$$

Minimizing first with respect to $\boldsymbol{\rho}$, with $\boldsymbol{\theta}$ being fixed, we obtain

$$\hat{\boldsymbol{\rho}} = (\mathbf{B}(\boldsymbol{\theta})^H \mathbf{B}(\boldsymbol{\theta}))^{-1} \mathbf{B}(\boldsymbol{\theta})^H \hat{\mathbf{y}}_1 \quad (23)$$

and hence, substituting this back into (20) the directions of the reflection are given by

$$\hat{\boldsymbol{\theta}} = \arg \min_{\boldsymbol{\theta}} \|\mathbf{P}_{\mathbf{B}(\boldsymbol{\theta})}^\perp (\hat{\mathbf{y}}_{m,1})\|^2 \quad (24)$$

where $\mathbf{P}_{\mathbf{B}(\boldsymbol{\theta})}^\perp$ is the projection on the orthogonal complement of the subspace spanned by the columns of $\mathbf{B}(\boldsymbol{\theta})$,

$$\mathbf{P}_{\mathbf{B}(\boldsymbol{\theta})}^\perp = \mathbf{I} - \mathbf{B}(\boldsymbol{\theta}) (\mathbf{B}^H(\boldsymbol{\theta}) \mathbf{B}(\boldsymbol{\theta}))^{-1} \mathbf{B}^H(\boldsymbol{\theta}) \quad (25)$$

Now using (3) we estimate $\hat{\mathbf{a}}_m = \mathbf{A}_l^{-1} \mathbf{y}_{ml}$.

4. THE ML ESTIMATOR

In this section we derive the Maximum Likelihood estimator for our problem. We show that this estimator is much more complicated than the LS estimator derived in the previous section. Let

$$\mathbf{y}_m = \begin{bmatrix} \mathbf{y}_{m,1} \\ \mathbf{y}_{m,2} \end{bmatrix} \quad (26)$$

and

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix} \quad (27)$$

From (3), the maximum likelihood estimator is given by

$$[\hat{\mathbf{a}}_1, \dots, \hat{\mathbf{a}}_p, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\rho}}] = \arg \min_{\mathbf{a}_1, \dots, \mathbf{a}_p, \boldsymbol{\theta}, \boldsymbol{\rho}} \sum_{m=1}^p \|\mathbf{y}_m - \mathbf{A} \mathbf{a}_m\|^2 \quad (28)$$

Minimizing first with respect to $\mathbf{a}_m$ we get

$$\hat{\mathbf{a}}_m = (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{y}_m \quad (29)$$

Now, from (27) and (6) we obtain

$$\mathbf{A}^H \mathbf{A} = \mathbf{F}(|\mathbf{W}_1|^2 + |\mathbf{W}_2|^2)^{-1} \mathbf{F}^H \quad (30)$$

where $\mathbf{W}_l = \text{diag}(\mathbf{F} \mathbf{w}_l)$. Substituting (30) and (6) into (29) yields

$$\hat{\mathbf{a}}_m = \mathbf{F} \mathbf{D} (\mathbf{W}_1^H \mathbf{F}^H \mathbf{y}_{m,2} + \mathbf{W}_2^H \mathbf{F}^H \mathbf{y}_{m,1}) \quad (31)$$

where

$$\mathbf{D} = (|\mathbf{W}_1|^2 + |\mathbf{W}_2|^2)^{-1} \quad (32)$$

Finally, substituting (31) and (6) into (28), and eliminating the left most $\mathbf{F}$ by Parseval identity, the resulting ML estimator becomes

$$[\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\rho}}] = \arg \max_{\boldsymbol{\theta}, \boldsymbol{\rho}} \sum_{m=1}^p \|\mathbf{W}_1 \mathbf{D} \mathbf{v}\|^2 + \|\mathbf{W}_2 \mathbf{D} \mathbf{v}\|^2 \quad (33)$$

where

$$\mathbf{v} = (\mathbf{W}_1^H \mathbf{F}^H \mathbf{y}_{m,2} + \mathbf{W}_2^H \mathbf{F}^H \mathbf{y}_{m,1}) \quad (34)$$

Notice that this estimator involves all the reflections parameters', i.e. the DOA's and the reflection coefficients, which are complex parameters, in contrast to the proposed LS estimator which involves only the reflections directions.

5. SIMULATION RESULTS

In this section we present the results of a simulated experiment that demonstrates the performance of the LS estimator. The array consisted of two sensors, 3.5λ apart, and the number of reflections was 2, i.e., $r_1 = r_2 = 2$. The relative angular separation in the two sets of measurements was $\theta_{1,2} = 15^\circ$ and $\theta_{2,2} = 30^\circ$, respectively. The signal to noise ration (SNR) varied from 20 dB to 40 dB. The reflection coefficients were $\rho_{1,1} = 1$, $\rho_{1,2} =$

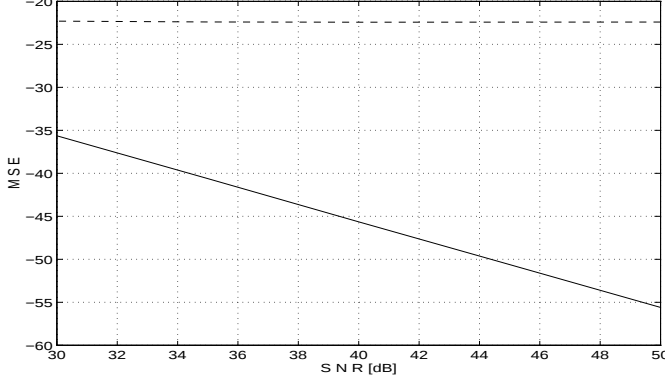


Figure 2. The dashed line : the error due to the multipath. The solid line : the error after the application of the algorithm. $\rho_{1,1} = 1, \rho_{1,2} = 0.05 + 0.01j, \rho_{2,1} = 1, \rho_{2,2} = 0.15 + 0.16j, \theta_{1,2} = 15^\circ$ and $\theta_{2,2} = 30^\circ$.

$0.05 + 0.01j, \rho_{2,1} = 1, \rho_{2,2} = 0.15 + 0.16j$. For each SNR we have performed 50 Monte-Carlo trials. Figure 2 presents the mean square error (MSE) of the steering vectors before and after the calibration algorithm. One can clearly see the improvement due to the algorithm. While the MSE due to the multipath is independent of SNR, the error after the application of the algorithm is greatly reduced.

In the second experiment, the relative angular separation between the reflections in the first set of measurements was held fixed at 15° , while the relative angular separation in the second set of measurements varied from 17° to 72° . The reflection coefficients were $\rho_{1,1} = 1, \rho_{1,2} = 0.11 + 0.2j, \rho_{2,1} = 0.7, \rho_{2,2} = 0.055 + 0.2j$ and the SNR was 30 dB. The MSE of the array manifolds for each set of measurements is presented in figure 3. Notice that the performance of the algorithm is essentially independent of the angular separation.

6. CONCLUSIONS

We have presented a novel method for the calibration of sensor arrays in the presence of multipath. The method is based on measuring the array manifold from two angularly separated locations, and involves a solution of a multidimensional optimization. The method does not depend on the relative angular locations of the reflections. A Maximum Likelihood estimator for the problem was computed as well, demonstrating the improved computational complexity of our LS estimator.

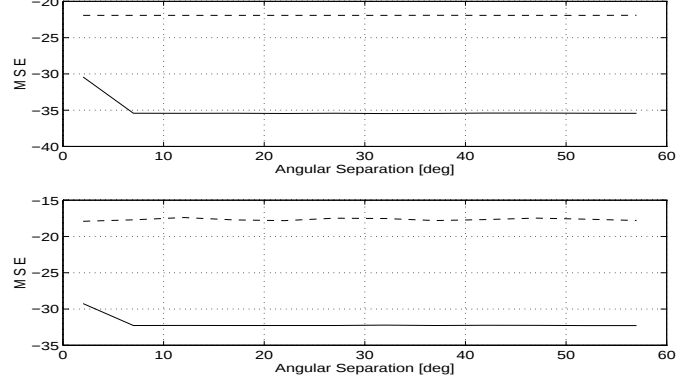


Figure 3. The dashed line : the error due to the multipath. The solid line : the error after the application of the algorithm. $\rho_{1,1} = 1, \rho_{1,2} = 0.11 + 0.2j, \rho_{2,1} = 0.7, \rho_{2,2} = 0.055 + 0.2j, \text{SNR}=30\text{dB}$.

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