THE MSE PERFORMANCE OF CONSTANT MODULUS RECEIVERS

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Abstract

The constant modulus algorithm (CMA) is an effective technique for blind receiver design in practice. Treating CMA as a linear estimation problem, effects of noise and channel conditions are investigated. For the class of channels with arbitary finite impulse responses, an analytical description of locations of constant modulus receivers and an upper bound of their mean squared errors (MSE) are derived. We show that, with proper initializations, CMA can achieve almost the same performance as the (nonblind) minimum mean square error (MMSE) receiver. Our analysis reveals a strong relationship between the (blind) constant modulus and the (nonblind) MMSE receivers. It also highlights the significance of initialization/reinitialization schemes. The approach developed in this paper also applies to CMA blind beamforming in array signal processing.

1. INTRODUCTION

Blind equalization is becoming a useful receiver design technique in some advanced digital communication systems. When applied in practice, it is important to compare its performance with nonblind receivers. A quantitative measure of performance degradation is particularly valuable.

One successful blind equalization scheme is the constant modulus algorithm (CMA). Performance of CMA has been studied by many researchers, but mostly for the local minimum in the noiseless case [3, 5, 1]. Effects of noise has been investigated recently. Since the CMA involves the optimization with a nonconvex cost function, theoretical analysis without approximation is difficult. A number of approaches [2, 6] have been presented based on various approximations at high signal to noise ratio. It is often not clear when such analysis is accurate.

This work was supported in part by the National Science Foundation under Contract NCR-9321813 and by the Office of Naval Research under Contract N00014-96-1-0895 and the CRASP.

Without involving approximation, a new geometrical technique [9, 11] has been proposed recently to address some important issues such as the existence of finite-length fractionally-spaced CMA local minimum and its MSE in the presence of noise. Unfortunately, one of the assumptions in [9, 11] is the invertibility (nonsingular) of the channel matrix. This excludes some very important practical applications such as T-spaced equalization, fractionally-spaced equalization with insufficient equalizer length.

This paper derives the MSE of constant modulus receiver under Gaussian noise for an *arbitrary* channel matrix. Although the method in [9, 11] can not be applied for a singular channel, the basic concept is still valid. In particular, the MSE analysis in this paper relates the constant modulus receiver to the MMSE (the optimal linear) receiver. The main results include (i) a sufficient condition for the existence of a constant modulus (CM) receiver in the neighborhood of a MMSE receiver; (ii) an analytical description of the region that contains a CMA local minimum in the neighborhoods of a MMSE receiver; (iii) an upper bound of the MSE of these constant modulus receivers. Using the derived MSE bound, we study the local minimum problem. We show that there exists a class of CMA local minima associated with the MMSE receiver with different delays. We also apply the result to the Ding's example of local minima presented [1]. The analysis shows that Ding's local minimum is in fact related to the MMSE equalizer.

2. PROBLEM FORMULATION

Consider a linear time-invariant system shown in Figure 1.

$$\mathbf{x} = \mathbf{H}\mathbf{s} + \mathbf{w}, \qquad (1)$$

$$y = \mathbf{f}^t\mathbf{x} = \mathbf{q}^t\mathbf{s} + \mathbf{f}^t\mathbf{w}, \qquad (2)$$

where $\mathbf{H}$ is an $n \times m$ channel matrix, $(\cdot)^t$ denotes transpose. This model is valid for both T-spaced and fractionally spaced FIR equalization. It also includes the beamforming problem in array signal processing. We assume that (A1) entries of $\mathbf{s}$ are zero mean, i.i.d. random variables with unit variance and the dispersion ratio $r = E\{|s|^4\}/E\{|s|^2\}$; (A2) entries of $\mathbf{w}$ are i.i.d. Gaussian random variables with variance $\sigma^2 > 0$.

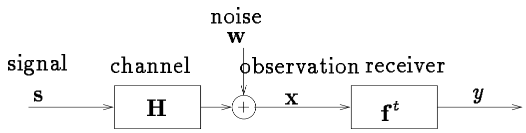

Figure 1: A linear estimation in a time invariant system.

The main objective of this paper is to determine the location and MSE of constant modulus (CM) receivers, and to show the connection between CM and MMSE receivers. Note that we do not assume $\mathbf{H}$ is full column rank, *i.e.,* $\mathbf{H}$ may be singular.

3. THE MSE OF CM RECEIVERS

3.1. An Equivalent Cost Function

The CM receiver minimizes the following objective function

$$\begin{aligned} J_c(\mathbf{f}) &\triangleq E\{(|y|^2 - r)^2\} \\ &= 3||\mathbf{f}||^4_{\mathbf{R}} - 2r||\mathbf{f}||^2_{\mathbf{R}} - (r-3)||\mathbf{H}^t\mathbf{f}||^4_4 + r^2 \end{aligned} \quad (3)$$

where $||\mathbf{f}||^2_{\mathbf{R}} = \mathbf{f}^t\mathbf{R}\mathbf{f}$, $||\cdot||_4$ denotes the 4-th norm, and

$$\mathbf{R} \triangleq E\{\mathbf{x}\mathbf{x}^t\} = \mathbf{H}\mathbf{H}^t + \sigma^2\mathbf{I}_n. \quad (4)$$

One important property is that the CM receiver must be in the "signal subspace" spanned by the columns of $\mathbf{H}$ [11]. Thus we can analyze CM receivers in terms the combined channel-receiver $\mathbf{q} \triangleq \mathbf{H}^t\mathbf{f}$, where $\mathbf{H}$ defines a 1-1 mapping between the column space and row space of $\mathbf{H}$. For $\mathbf{q} \in Row(\mathbf{H})$, define

$$\begin{aligned} J(\mathbf{q}) &\triangleq J_c((\mathbf{H}^t)^\dagger\mathbf{q}) \\ &= 3||\mathbf{q}||^4_{\mathbf{\Phi}} - 2r||\mathbf{q}||^2_{\mathbf{\Phi}} - (3-r)||\mathbf{q}||^4_4 + r^2, \end{aligned} \quad (5)$$

where

$$\mathbf{\Phi} = \mathbf{I}_m + \sigma^2\mathbf{H}^\dagger(\mathbf{H}^\dagger)^t. \quad (6)$$

Therefore, we have

$$\min J_c(\mathbf{f}),\quad \mathbf{f} \in Col(\mathbf{H}) \quad \Leftrightarrow \quad \min J(\mathbf{q}),\quad \mathbf{q} \in Row(\mathbf{H}).$$

3.2. A Geometrical Approach

A key step is to bound the location of constant modulus receivers (local minima of the constant modulus cost function). Although the method in [9, 11] can not be applied for singular channel, the basic concept is still valid. In contrast to the nonsingular case, we encounter a more difficult *constrained* optimization problem as shown in Figure 2. Let $\mathcal{S}$ be the linear subspace spanned by the row space of $\mathbf{H}$, and we have mentioned earlier that all CMA receivers are within $\mathcal{S}$. Suppose that there is a region $\mathcal{B}$ with boundary $\partial\mathcal{B}$ and $\mathbf{q}_r$ is a reference point in $\mathcal{B}\bigcap\mathcal{S}$. If the cost $J(\mathbf{q})$ on the boundary $\partial\mathcal{B}\bigcap\mathcal{S}$ is greater than that of the reference $\mathbf{q}_r$, then there exists at least a minimum of $J(\mathbf{q})$ in $\mathcal{B}\bigcap\mathcal{S}$.

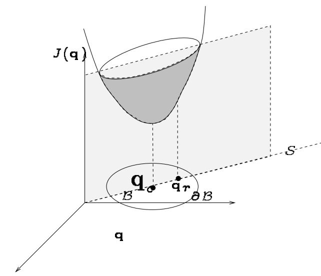

Figure 2: The geometrical approach with subspace constraint.

Our approach includes following three steps:

Step I: Reference

We choose the reference as the vector which is colinear with the MMSE receiver and has minimum CM cost. Specifically, the MMSE receiver for estimating s_ν is given by

$$\mathbf{q}_m = \mathbf{H}^t\mathbf{R}^{-1}\mathbf{H}\mathbf{e}_\nu, \quad (7)$$

where $\mathbf{e}_\nu$ is a unit column vector with 1 at the νth entry and zeros elsewhere. Define the reference $\mathbf{q}_r = \alpha_r\mathbf{q}_m$, where α_r minimizes the CM cost function (5):

$$\alpha_r = \arg\min_\alpha J(\alpha\mathbf{q}_m) = \sqrt{\frac{r||\mathbf{q}_m||^2_{\mathbf{\Phi}}}{3||\mathbf{q}_m||^4_{\mathbf{\Phi}} - (3-r)||\mathbf{q}_m||^4_4}}.$$

Thus, we obtained the reference $\mathbf{q}_r$ and its CM cost $J(\mathbf{q}_r)$.

Step II: Cone-type Region

The neighborhood is defined according to the receiver gain θ and the extra unbiased mean square error (UMSE) u. For a given receiver $\mathbf{q} = [q_1 \cdots q_{\nu-1}\ q_\nu\ q_{\nu+1} \cdots q_m]^t$ whose output y is the estimate of s_ν, the receiver gain and interference is defined by

$$\theta \triangleq q_\nu = \mathbf{e}^t_\nu\mathbf{q} \qquad \mathbf{q}_I \triangleq [q_1 \cdots q_{\nu-1}\ q_{\nu+1} \cdots q_m]/q_\nu, \quad (8)$$

If we scale the receiver $\mathbf{q}$ by $\frac{1}{\theta}$, we obtain the (conditionally) unbiased receiver u, *i.e.,* $u = \frac{y}{\theta}$. Based on θ and the MSE of u, we define the neighborhood as

$$\{\mathbf{q} \in Row(\mathbf{H}) : \theta_L \le \theta \le \theta_U,\ MSE(u) - MSE(u_m) \le \delta^2_U\} \quad (9)$$

where u_m is the unbiased MMSE receiver. It is shown in [10] that the neighborhood (9) is equivalent to a sliced cone given by

$$\mathcal{B} = \{\mathbf{q} \in Row(\mathbf{H}) : \theta_L \leq \theta \leq \theta_U, ||\mathbf{q}_I - \mathbf{q}_{mI}||_{\mathbf{C}} \leq \delta_U\} \tag{10}$$

where matrix $\mathbf{C}$ is the submatrix of $\mathbf{\Phi}$ by deleting the νth column and νth row.

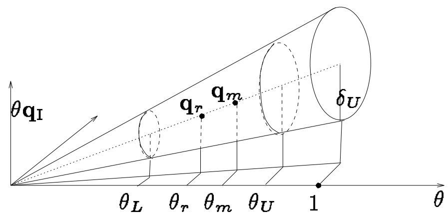

Figure 3: A cone-type region.

Step III: CM Cost on the Boundary

Based on the cone-type region, the CM cost function $J(\mathbf{q})$ can be reduced to a function in terms of gain θ and extra UMSE δ_U^2 [10].

Lemma 1 *Let $\mathbf{q}_o = \mathbf{\Phi}^{-1}\mathbf{e}_\nu$, θ_o and $\mathbf{q}_{o\mathrm{I}}$ are the gain and interference of $\mathbf{q}_o$ respectively. For any $\mathbf{q} \in \mathcal{B}$,*

$$J(\mathbf{q}) - J(\mathbf{q}_r) \geq c_2(\delta)\theta^4 + c_1(\delta)\theta^2 + c_0, \quad \text{(equality holds iff } \delta = 0\text{)} \tag{11}$$

where

$$\begin{aligned} \delta &\triangleq \sqrt{\delta_U^2 + \delta_0^2}, \quad \delta_0 \triangleq ||\mathbf{q}_{m\mathrm{I}} - \mathbf{q}_{o\mathrm{I}}||_{\mathbf{C}}, \\ c_0 &\triangleq r^2 - J(\mathbf{q}_r), \quad c_1(\delta) \triangleq -2r(\delta^2 + \frac{1}{\theta_o}), \\ c_2(\delta) &\triangleq 3(\delta^2 + \frac{1}{\theta_o})^2 - (3-r)(1 + (\delta + ||\mathbf{q}_{o\mathrm{I}}||_4)^4). \end{aligned}$$

3.3. Location and MSE of CM Receivers

From Lemma 1, we can see that the $J(\mathbf{q}) - J(\mathbf{q}_r)$ is lower bounded by a second-order polynomial of θ^2 with coefficients $c_2(\delta), c_1(\delta)$, and c_0, all of which are functions of δ but not of θ. Thus the region $\mathcal{B}$ containing CM receivers can be obtained by choosing θ_L, θ_U, and δ_U such that $J(\mathbf{q}) - J(\mathbf{q}_r) > 0$ for all $\mathbf{q} \in \partial\mathcal{B}$.

Theorem 1 *Given $\mathbf{H}$, r, σ^2 and ν. Let $D(\delta) \triangleq c_1^2(\delta) - 4c_2(\delta)c_0$. If (1) $D(\delta)$ has real roots in (δ_0, ∞), and the smallest of which is δ_*; (2) $\forall\delta \in [\delta_0, \delta_*]$, $c_2(\delta) > 0$, then there exists a local minimum in the region (10) and*

$$\delta_U^2 = \delta_*^2 - \delta_0^2 \tag{12}$$

$$\theta_L = \min_{\delta_0 \leq \delta \leq \delta_*} \sqrt{\frac{-c_1(\delta) - \sqrt{c_1^2(\delta) - 4c_2(\delta)c_0}}{2c_2(\delta)}} \tag{13}$$

$$\theta_U = \max_{\delta_0 \leq \delta \leq \delta_*} \sqrt{\frac{-c_1(\delta) + \sqrt{c_1^2(\delta) - 4c_2(\delta)c_0}}{2c_2(\delta)}} \tag{14}$$

Once $\delta_U, \theta_L, \theta_U$ are obtained, the MSE upper bound of CM receivers in this region can be obtained. Furthermore, the MSE of the reference can be used as an approximation [10].

Theorem 2 *The MSE of CM receivers in $\mathcal{B}$ is upper bounded by $\mathcal{E}_U$ and is approximated by $\hat{\mathcal{E}}$:*

$$\begin{aligned} \mathcal{E}_U &= \max\{\frac{(\theta_U - \theta_o)^2}{\theta_o} + (\theta_U\delta_*)^2, \frac{(\theta_L - \theta_o)^2}{\theta_o} + (\theta_L\delta_*)^2\} \\ &\quad +1 - \theta_o \qquad (15) \\ \hat{\mathcal{E}} &= \frac{(\theta_r - \theta_o)^2}{\theta_o} + (\theta_r\delta_0)^2 + 1 - \theta_o, \qquad (16) \end{aligned}$$

4. BAUD-RATE EQUALIZATION

We apply the analysis above to baud-rate equalization where the channel matrix is singular. For a baud-rate equalizer of a finite impulse response channel $\mathbf{h} = [h_0, \cdots, h_{L-1}]^t$, the corresponding vector representation in Figure 1 is given by

$$\begin{aligned} \mathbf{H} &\triangleq \begin{pmatrix} h_0 & \cdots & h_L & & \\ & \ddots & & \ddots & \\ & & h_0 & \cdots & h_L \end{pmatrix}_{n\times m}, \\ \mathbf{x} &\triangleq [x(k), \cdots, x(k-n-1)]^t, \\ \mathbf{s} &\triangleq [s(k), \cdots, x(k-n-L)]^t, \quad m = n + L - 1. \end{aligned}$$

4.1. Equalizer Delay and Ill Convergence

We consider now an example [9] where the channel impulse response is {0.0113, -0.0285, 0.0606, 0.8701, 0.4392, 0.4392}. We compare $\mathcal{E}_U$ and $\hat{\mathcal{E}}$ with the actual MSE of CM ($\mathcal{E}_c$) and the MMSE ($\mathcal{E}_m$). The CM equalizers are obtained from the gradient search for the local minima initialized at MMSE receivers.

In equalization, $\nu - 1$ represents the delay of the combined channel and equalizer response. Due to the nature of blind estimation, the delay can not be specified in the CM equalizer. Depending on the initialization, the CMA may converge to any local minimum. From Figure 4, the CM equalizer at $\nu = 11$ has $10dB$ MSE loss comparing with the optimal CM equalizer at $\nu = 5$.

4.2. Ding's Example

For finite-length T-spaced CM equalizer, Ding showed the existence of local minimum for AR channels [1]. In this section, we show that Ding's local minimum in fact belongs to the local minima associated with MMSE equalizers. Consider the following AR channel

$$x(k) + \alpha x(k-1) = s(k). \tag{17}$$

The equivalent channel impulse response is $h_i = (-\alpha)^i$. For a 2-tap equalizer $[f_1, f_2]^t$, Ding showed that there

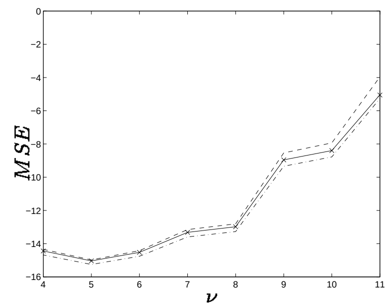

Figure 4: Dashed line: $\mathcal{E}_U$; Solid line: $\mathcal{E}_c$; Cross: $\hat{\mathcal{E}}$; Dashdot: $\mathcal{E}_m$; $SNR = 20dB$, $r = 1$(BPSK), $n = 8$.

exists two local minima given by

$$\mathbf{f}_c^{(1)} = [1, \alpha]^t \quad (18)$$

$$\mathbf{f}_c^{(2)} = [0, \sqrt{\frac{1-\alpha^4}{3(1+\alpha^2) - 2r(1-\alpha^2)}}]^t. \quad (19)$$

According to Theorem 1, it is interesting to examine the relationship between these local minima and the MMSE solutions. It can be shown that the channel matrix and the covariance of $\mathbf{x} = [x(k), x(k-1)]^t$ in (17) are given by

$$\mathbf{H} = \begin{pmatrix} 1 & -\alpha & \alpha^2 & \cdots \\ 0 & 1 & -\alpha & \cdots \end{pmatrix}$$

$$\mathbf{R} \triangleq E\{\mathbf{x}\mathbf{x}^t\} = \frac{1}{1-\alpha^2}\begin{pmatrix} 1 & -\alpha \\ -\alpha & 1 \end{pmatrix}.$$

The MMSE receiver at $\nu = 1, 2$ are given by

$$\mathbf{f}_m^{(1)} = \mathbf{R}^{-1}\mathbf{H}(:,1) = [1, \alpha]^t = \mathbf{f}_c^{(1)}$$

$$\mathbf{f}_m^{(2)} = \mathbf{R}^{-1}\mathbf{H}(:,2) = [0, 1-\alpha^2]^t \propto \mathbf{f}_c^{(2)}.$$

When $r = 1$ and $\alpha = 0.5$, $\mathbf{f}_c^{(2)} = [0, 0.6455]^t$ which is close to $\mathbf{f}_m^{(2)} = [0, 0.75]^t$.

We also apply Theorem 1 to this example with the channel truncated to $L_h = 20$. Using (10), $\mathbf{f}_c^{(2)}$ is located in the neighborhood of $\mathbf{f}_m^{(2)}$ and the MSE bound are given in Table 1. In this case, the CM equalizer is colinear with the MMSE receiver. The estimate MSE ($\hat{\mathcal{E}}$) turns out to be the exact MSE of the CMA ($\mathcal{E}_c$).

ν	$\mathcal{E}_U$	$\hat{\mathcal{E}}$	$\mathcal{E}_c$	$\mathcal{E}_m$
1	0.0000	0.0000	0.0000	0.0000
2	0.2983	0.2646	0.2646	0.2500

Table 1: MSE of Ding's example. $r = 1$, $\alpha = 0.5$.

5. CONCLUSION

In this paper, the MSE performance of the CMA is investigated for arbitrary (real) channels including baud-rate equalization and fractionally-spaced equalization. Given the channel matrix, the dispersion ratio and the noise variance, the location and MSE bound of a CM receiver in the neighborhood of a MMSE receiver is derived analytically. The analysis shows that, while in some cases the CM equalizer performs almost as well as the (nonblind) MMSE receiver, it is also possible that, due to its blind nature, CMA may perform considerably worse than a (nonblind) MMSE receiver. Our results underscore the importance of developing initialization and reinitialization schemes for CMA, one of which is presented in [7].

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