A FOUR-PARAMETER ATOMIC DECOMPOSITION OF CHIRPLETS

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ABSTRACT

A new four-parameter atomic decomposition of chirplets is developed for compact representation of signals with chirp components. The four-parameter atom is obtained by scaling the Gaussian function, and then applying the fractional Fourier transform (FRFT), time-shift and frequency-shift operators to the scaled Gaussian. The decomposition is realized by extending the matching pursuit algorithm to four parameters. For this purpose, the four-parameter space is discretized to obtain a dense subset in the Hilbert space. Also, a related time-frequency distribution is developed for clear visualization of the signal components. The decomposition provides a more compact and precise representation of chirp components as compared to the three-parameter ones.

1. INTRODUCTION

One of the main research interests in signal processing is to decompose signals into well-defined and localized components in the time-frequency plane, which are called time-frequency atoms. Also, a clear and readable display of the energy distribution of a signal is desired. Various decompositions with atoms up to three parameters have been used in the literature such as the Gabor expansion, wavelet transform [1] and the wavelet packets, [2], [3]. However, these tools fail to represent chirp-like components in a compact and precise way.

Recently, Mann and Haykin have used the time-shift, frequency-shift, scale, frequency-shear (chirp multiplication) and the time-shear (chirp convolution) operators to construct the chirplet transform for clear analysis of the chirp components [4]. Two-parameter subgroups of this transform were studied in [5].

In this paper, we have used the rotation operator in the time-frequency plane, to extend the three-parameter decompositions for compact representation of chirp-like components. Since the rotation corresponds to an orthogonal transformation of the time-frequency plane, which is shape-preserving, it is a more direct extension of the three parameters. Rotation is performed by using the fractional Fourier transform (FRFT) operator [6]. By applying the scaling, rotation, time-shift and frequency-shift operators to the Gaussian function, the four-parameter chirplet atom is obtained.

A decomposition that expands any function in terms of these four-parameter atoms, is developed. The decomposition is realized by adapting the matching pursuit algorithm [3] to four parameters. The four-parameter space is carefully discretized to obtain a complete discrete set which is suitable for efficient computation. Also, a time-frequency distribution (TFD) that provides a clear and readable display of the decomposition is developed.

2. ATOMIC DECOMPOSITION OF CHIRPLETS

For local analysis of signals, the set of time-frequency atoms should be highly concentrated in the time-frequency plane. Since the Gaussian occupies the minimum area in the time-frequency plane, it is used as an elementary atom. Applying a series of area-preserving operators to the Gaussian function, the chirplet atom is obtained.

Rotation in the time-frequency plane is achieved by using the FRFT, which is defined as [6]

$$\begin{aligned} f_{-\alpha}(t) &\triangleq (\Gamma_{-\alpha} f)(t) \\ &\triangleq \sqrt{\frac{1-j\cot\alpha}{2\pi}}\, e^{j\frac{\cot\alpha}{2}t^2} \int_{-\infty}^{\infty} f(\tau) e^{j\frac{\cot\alpha}{2}\tau^2} \\ &\quad \times e^{-j\csc\alpha\tau t} d\tau \end{aligned} \tag{1}$$

where Γ_α is the rotation operator. The minus sign in $\Gamma_{-\alpha}$ is used to define the counter-clockwise direction in the time-frequency plane as the positive direction of rotation. Use of the rotation operator results in a confusion of manipulating the time and frequency variables, which have different units, as having the same unit. To solve this problem, dimensional normalization of the

time-frequency plane is developed in [7]. Throughout this paper, it is assumed that, this dimensional normalization was performed, and the time-frequency axes have the units of $\sqrt{rad}$.

The Gaussian, $g(t) = \frac{1}{\pi^{1/4}} e^{-\frac{t^2}{2}}$, is scaled to obtain $g_s(t) \triangleq \frac{1}{\sqrt{s}} g(\frac{t}{s})$. The rotated atom is obtained by applying the rotation operator to the scaled Gaussian. Then, the time and frequency shifts are applied successively to obtain the four-parameter chirplet atom as

$$\begin{aligned} g_\beta(t) &\triangleq (\Gamma_\alpha g_s)(t-u)e^{j\nu t} \\ &\triangleq g_{s,\alpha}(t-u)e^{j\nu t} \end{aligned} \quad (2)$$

where $\beta \triangleq (s, \alpha, u, \nu)$ is the index of the atom. The set of indexes Υ, are chosen such that the resulting set of atoms are unique,

$$\Upsilon \triangleq \begin{cases} (1,\infty) \times (-\pi/2, \pi/2) \times \mathbf{R^2}, & \alpha \neq 0 \\ \mathbf{R^+} \times \mathbf{R^2}, & \alpha = 0. \end{cases} \quad (3)$$

The scaled and rotated atom $g_{s,\alpha}(t)$ is found as

$$\begin{aligned} g_{s,\alpha}(t) = &\frac{\sqrt{s}e^{jc_1}}{\pi^{1/4}(\sin^2\alpha + s^4\cos^2\alpha)^{1/4}} \\ &\times \exp\{-\frac{s^2 - j(s^4-1)\cos\alpha\sin\alpha}{2(\sin^2\alpha + s^4\cos^2\alpha)}t^2\} \end{aligned} \quad (4)$$

where $c_1 = \frac{\pi}{4} - \frac{\alpha}{2} - \frac{\arctan(s^2\cot\alpha)}{2}$. Atoms are normalized, i.e., $||g_\beta(t)|| = 1$. The effects of these operations to the Wigner distribution (WD) of the basic atom are shown in Fig. 1. From this figure it is obvious that, s shows the window size when $\alpha = 0$. When rotation occurs ($\alpha \neq 0$), s is a measure of the directivity (d_β) of the Gaussian atom. Directivity can be thought as the aspect ratio of the Gaussian blob and defined as $d_\beta \triangleq s^2$. The parameters α, u and ν indicate the orientation, time-center and frequency-center of the atom in the time-frequency plane, respectively. Rotation angle is a measure of the chirp-rate of a local signal component.

The set of functions, which is called the *dictionary*, $\mathcal{D} \triangleq \{g_\beta(t)\}_{\beta\in\Upsilon}$ is complete [1] in the Hilbert space. Since the set is highly redundant, an appropriate countable subset of the atoms $\{g_{\beta_n}(t)\}_{n\in\mathbf{N}}$ with $\beta_n = (s_n, \alpha_n, u_n, \nu_n)$ must be selected such that any signal $f(t)$ can be written as

$$f(t) = \sum_{n=0}^{\infty} a_n g_{\beta_n}(t). \quad (5)$$

[1] It is shown in [8] that the set of three-parameter wavelet packets is complete. Since the three-parameter dictionary is a subset of the chirplet dictionary, the four-parameter set is complete, as well.

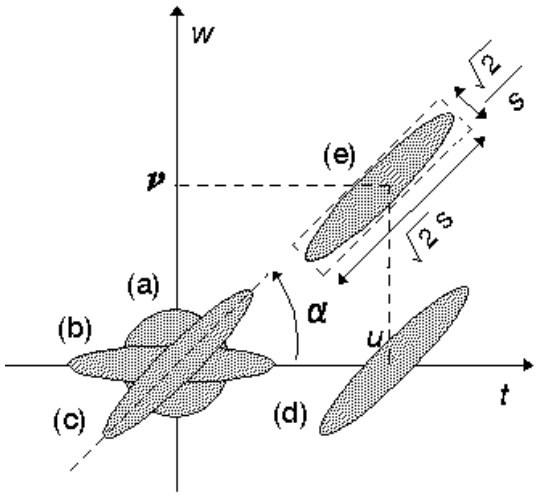

Figure 1: Change of the WD of an atom with parameters: (a) $W_g(t,\omega)$, (b) $W_{g_s}(t,\omega)$, (c) $W_{g_{s,\alpha}}(t,\omega)$, (d) $W_{g_{s,\alpha}}(t-u,\omega)$, (e) $W_{g_\beta} = W_{g_{s,\alpha}}(t-u,\omega-\nu)$.

In order to realize this decomposition the matching pursuit algorithm [3] is used. Matching pursuit is an iterative algorithm that selects an element at each iteration from a dictionary of atoms $\mathcal{D}$ to best match the inner structures of a signal, and finds its corresponding coefficient.

Although the algorithm is defined for three-parameters originally, it can be extended to larger dictionaries, provided that some constraints are satisfied. First, $\mathcal{D}$ must be a complete set, which is shown above. Second, atoms must be normalized (unit energy) and have a decay of $O(\frac{1}{t^2+1})$ [3]. By checking equation (4), it can be easily shown that this is satisfied. The main issue in the adaptation of the dictionary to the matching pursuit algorithm is to obtain a discrete sub-dictionary $\mathcal{D}_\mu$ [3], which satisfies

$$\sup_{\beta\in\Upsilon_\mu} |< f, g_\beta >| \geq \mu \sup_{\beta\in\Upsilon} |< f, g_\beta >|, \quad 0 < \mu \leq 1. \quad (6)$$

where Υ and Υ_μ are the continuous and the discrete index sets, respectively. The discrete index set is defined in the next section, where $\beta \in \Upsilon_\mu \Leftrightarrow g_\beta(t) \in \mathcal{D}_\mu$. The set Υ_μ is generally much smaller than Υ, so that the search time for the optimum index is limited efficiently.

3. THE DISCRETE DICTIONARY

In [3], a three-parameter discrete set is developed, that is the ensemble of Gabor lattices indexed by the scale parameter. In this section, the discrete four-parameter set that satisfies (6) is defined. This set is a family of rotated three-parameter lattices as shown in Fig.2.

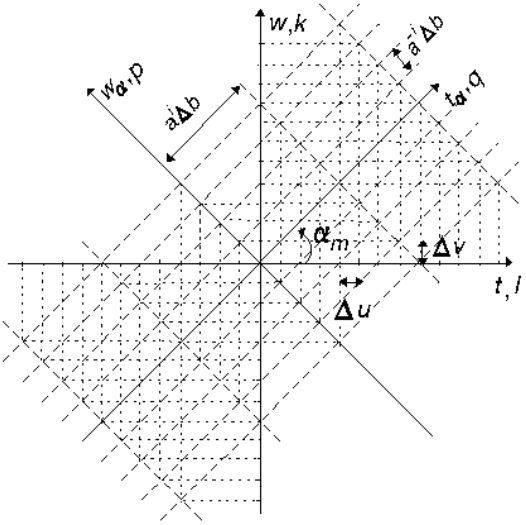

Figure 2: A lattice mapping for $a^i = 2$ and $\alpha_m = 45°$.

We define the discrete four-parameter index set as $\Upsilon_\mu \triangleq (a^i, \alpha_m, l(q,p)\Delta u, k(q,p)\Delta\nu)$ where $a > 1, i, l(q,p)$, $k(q,p), m, q$ and p are all integers. The discrete angles are given by

$$\alpha_m \triangleq \begin{cases} \arctan(\frac{m}{a^{2r}}), & r \geq 0, m \in I_1 \\ \frac{\pi}{2} - \arctan(\frac{m_1}{a^{2r}}), & r \geq 0, m \in I_2 \\ -\frac{\pi}{2} + \arctan(\frac{m_2}{a^{2r}}), & r \geq 0, m \in I_3 \\ 0, & r < 0 \end{cases} \quad (7)$$

where $r \triangleq i - i_0$ is the relative scale index with $i_0 \in Z^+$, and $m_1 = 2a^{2r} - m, m_2 = 2a^{2r} + m$. The intervals are defined as $I_1 = [-a^{2r}, a^{2r}], I_2 = (a^{2r}, 2a^{2r})$ and $I_3 = (-2a^{2r}, -a^{2r})$. The fixed scale index i_0 specifies the scale where the rotation begins. Time and frequency discretization step lengths and their corresponding indexes are given respectively as

$$\Delta u = \begin{cases} (\Delta b a^{-i})/(\sqrt{m^2 + a^{4r}}), & m \in I_1 \\ (\Delta b a^{(i-2i_0)})/(\sqrt{{m_1}^2 + a^{4r}}), & m \in I_2 \\ (\Delta b a^{(i-2i_0)})/(\sqrt{{m_2}^2 + a^{4r}}), & m \in I_3 \end{cases}$$

$$\Delta\nu = \begin{cases} (\Delta b a^{(i-2i_0)})/(\sqrt{m^2 + a^{4r}}), & m \in I_1 \\ (\Delta b a^{-i})/(\sqrt{{m_1}^2 + a^{4r}}), & m \in I_2 \\ (\Delta b a^{-i})/(\sqrt{{m_2}^2 + a^{4r}}), & m \in I_3 \end{cases}$$

$$l(q,p) = \begin{cases} a^{2(2i-i_0)}q - mp, & m \in I_1 \\ a^{2i_0}m_1 q - p, & m \in I_2 \\ a^{2i_0}m_2 q + p, & m \in I_3 \end{cases}$$

$$k(q,p) = \begin{cases} a^{2i_0}mq + p, & m \in I_1 \\ a^{2(2i-i_0)}q + m_1 p, & m \in I_2 \\ -a^{2(2i-i_0)}q + m_2 p, & m \in I_3 \end{cases}$$

where $\Delta b, \Delta u, \Delta\nu \in \mathbf{R}^+$.

As seen from Fig. 2, α_m's are the discrete angles that map lattices in the rotated coordinates to the lattices in the Cartesian coordinates. The points $(l(q,p)\Delta u, k(q,p)\Delta\nu)$ in the Cartesian coordinates are the projections of the lattice points $(qa^i\Delta b, pa^{-i}\Delta b)$ in the rotated coordinates. The discrete step length Δb is selected as $\Delta b = \sqrt{\frac{2\pi}{R}}$, where $R > 1$ is the redundancy ratio [3]. The following theorem verifies that this discrete parameter set satisfies the required constraint.

Theorem 1 *Let the rotation occurs only for $i \geq i_0 > 0$. Let $\Upsilon_\mu = (a^i, \alpha_m, l(q,p)\Delta u, k(q,p)\Delta\nu)$ which is defined above, be the discrete set of all indices β that is a subset of Υ defined in (3). Then there exist a constant $\mu > 0$ such that for all $f(t) \in \mathbf{L^2}(\mathbf{R})$*

$$\sup_{\beta \in \Upsilon_\mu} |< f, g_\beta >| \geq \mu \sup_{\beta \in \Upsilon} |< f, g_\beta >| . \quad (8)$$

The proof [2] of the theorem is given in [9].

4. THE TFD AND THE IMPLEMENTATION

Using the matching pursuit algorithm with the four-parameter discrete dictionary, any signal $f(t)$ can be decomposed as in (5). The three-parameter TFD defined in [3] can be extended to the four-parameter case by adding the weighted WDs of the atoms in (5) as shown below

$$E_f(t,\omega) \triangleq \sum_{n=0}^{\infty} |a_n|^2 W_{g_{\beta_n}}(t,\omega). \quad (9)$$

Since the matching pursuit is an energy-preserving decomposition [3], this TFD also preserves energy. The TFD, that is positive and free of interference terms, provides a clear and readable picture in the time-frequency plane.

The number of angles used in the discrete dictionary $\mathcal{D}_\mu$ can be limited by specifying a minimum angular resolution which is independent of the signal length. The numerical complexity of the four-parameter decomposition is higher than the three-parameter one by a factor of at most the maximum number of discrete angles.

A synthetic signal, shown in Fig.3, is decomposed with the three-parameter and the four-parameter dictionaries. In Fig. 4 and Fig. 5, the three-parameter and the four-parameter TFDs of the signal are shown, respectively. Note that, the two chirps crossing each other are tracked with two rotated atoms in Fig. 5,

[2] The proof is the straightforward extension of the corresponding proof of the three-parameter one given in [3].

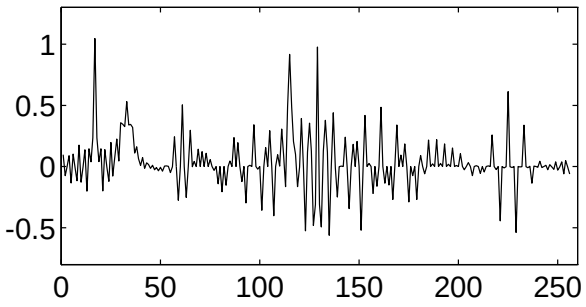

Figure 3: A synthetic signal composed of various time-frequency structures.

although the same components are represented with many atoms in Fig. 4, which results in a dilution of an information.

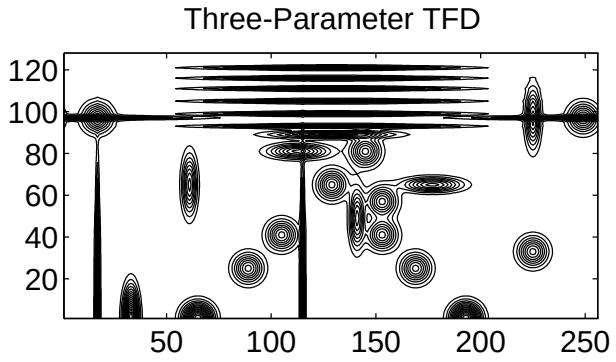

Figure 4: Three-parameter TFD of the signal in Fig.3. Contour plot of the TFD is displayed, where darker regions have higher energy.

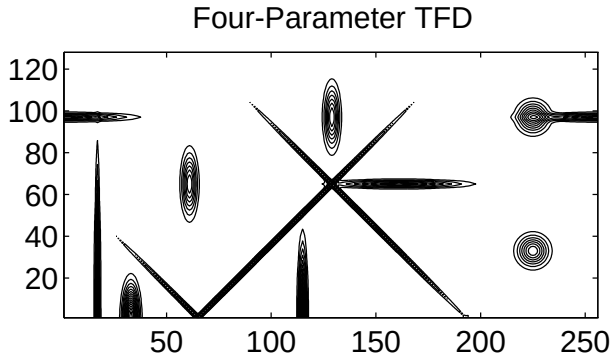

Figure 5: Four-parameter TFD of the reconstructed signal in Fig.3.

5. CONCLUSIONS

As seen from the above examples, the four-parameter decomposition provides a more compact and accurate representations of the chirp components as compared to the three-parameter ones. The decomposition is especially useful for chirp and instantaneous frequency analysis. The related TFD is used for a clear visualization of the signal components in terms of well-defined energy areas in the time-frequency plane. The time-frequency structures of complex signals can easily be identified with this TFD.

6. REFERENCES

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