ON THE USE OF HIGH ORDER AMBIGUITY FUNCTION FOR MULTICOMPONENT POLYNOMIAL PHASE SIGNALS

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ABSTRACT

Nonstationary signals appear often in real-life applications and many of them can be modeled as polynomial phase signals (PPS). High-order ambiguity function (HAF) was first introduced to estimate the parameters of a single component PPS. But due to its high nonlinearity, HAF has not been widely used for multi-component PPS which appear for example, in Doppler radar applications when multiple targets are tracked simultaneously. We present a theory in this paper that HAF is virtually additive for multi-component PPS and illustrate our findings with numerical simulations.

1. INTRODUCTION

Signals encountered in engineering applications such as communications, radar, and sonar often involve amplitude (AM) and/or frequency modulation (FM). An AM-FM signal can be written as $x(t) = \rho(t)e^{j\phi(t)}$, where $\rho(t)$ represents the time-varying amplitude and $\phi(t)$ stands for the phase. The instantaneous frequency is found as $f(t) = d\phi(t)/dt$. Although non-parametric techniques are available to track amplitude and frequency variations, we focus on parametric models here because they offer parsimony and inherently unlimited resolution.

The phase function of a large class of AM-FM processes can be modeled by a polynomial function of t. It is known that in active systems, radar echoes from maneuvering targets have nonlinear phase characteristics, which depend on the target trajectory. In Kelly [2], a radar echo is expressed as $x(t) = \rho(t)e^{jr(t)}$, and the trajectory is approximated by

$$r(t) \doteq r_0 + v_y t + \frac{1}{2}\left(a_y + \frac{v_x^2}{r_0}\right)t^2 + \frac{1}{2r_0}\, a_x v_x t^3 + \frac{1}{8}\, a_x^2 t^4 + \cdots . \tag{1}$$

From (1) we see that radial velocity v_y introduces a linear phase term in $x(t)$; radial acceleration a_y and cross-range velocity v_x induce a quadratic phase term; a_x and v_x contribute a cubic phase term, and so forth. The coefficients of the power series of $r(t)$ are thus related to the kinetic parameters of the moving target, cf. also Rihaczek [6]. According to the Stone-Weierstrass theorem, any continuous function (such as $r(t)$) over a closed interval can be approximated uniformly by a polynomial function. Therefore the class of PPS is rather broad.

Single-component PPS have been investigated extensively in recent years using the high-order ambiguity function (HAF), introduced by Peleg and Porat [3] (see e.g., [5, Ch. 12]). HAF has proven effective for parametric estimation of single-component PPS, and results on constant, random, as well as time-varying amplitude single-component PPS have appeared (see e.g., [10] and references therein).

However, signals arising from real life applications often have multiple components, and their estimation poses a great challenge. The effectiveness of HAF for single component PPS lies in the fact that it reduces a PPS of appropriate order to a line in the frequency domain. But since HAF is nonlinear, many cross-terms appear when it is applied to multicomponent PPS. These cross terms amount to the so-called "deterministic noise" effect. Issues such as how the magnitudes of such noise terms compare with the strengths of the signal peaks and whether they are narrowband or broadband have not been thoroughly investigated before and are being addressed in this paper.

In Section 2 of this paper, we review HAF and the related concepts for single component PPS. In Section 3, we focus on multi-component chirp signals (PPS of order 2) and examine the effect of the cross-terms. The results of Section 3 are then generalized in Section 4 to Mth-order multi-component PPS. Proofs of all theorems can be found in [9].

2. HAF AND SINGLE-COMPONENT PPS

HAF was originally devised to estimate the phase coefficients of a single-component constant amplitude PPS of order M,

$$y(t) = \rho\, e^{j\phi(t)} = \rho\, e^{j\sum_{m=0}^{M} a_m t^m}. \tag{2}$$

In this paper, we assume that the data are in discrete-time; i.e., $t = 0, 1, \ldots, T-1$. For integer $\tau \neq 0$, define $\mathcal{P}_2[y(t);\tau] = y(t)y^*(t-\tau)$, which can be viewed as a second-order instantaneous moment of $y(t)$. Since multiplying $y(t)$ by its conjugated and lagged copy $y^*(t-\tau)$ is equivalent to differencing in the phase of $y(t)$, it follows easily that $\mathcal{P}_2[y(t);\tau]$ is a new PPS of order $M-1$. The above operation can then be repeated to eventually reduce a PPS of any order to a complex constant. Each product process is called a *high-order instantaneous moment* (HIM) of $y(t)$. Its general form was first introduced by Peleg and Porat (see e.g., [5, Ch. 12]). We quote their results here for easy reference.

Let $y(t)$ be a complex valued signal, and define $y^{(*q)}(t) = y(t)$ for q even, $y^{(*q)}(t) = y^*(t)$ for q odd. The Mth-order HIM operator is defined as

$$\mathcal{P}_M[y(t);\tau] \triangleq \prod_{q=0}^{M-1} [y^{(*q)}(t-q\tau)]^{\binom{M-1}{q}}, \tag{3}$$

where $\binom{M-1}{q}$ is the binomial coefficient. For $y(t)$ of (2), we have

$$\mathcal{P}_M[y(t);\tau] = \rho^{2^{M-1}} e^{j\tilde{\omega}t+\tilde{\phi}}, \quad \tilde{\omega} \triangleq M!\,\tau^{M-1}\, a_M. \tag{4}$$

We notice that $\mathcal{P}_M[y(t);\tau]$ reduces the Mth-order PPS of (2) to a constant amplitude harmonic with amplitude

$\rho^{2^{M-1}}$, frequency $\tilde{\omega}$ and phase $\tilde{\phi}$ ($\tilde{\phi}$ is a function of the a_m's and is not of concern here). HIM of order $> M$ reduces an Mth-order PPS to a complex constant.

Since $\mathcal{P}_M[y(t);\tau]$ is a periodic sequence, we consider its Fourier series (FS) coefficient for $\alpha \in [-\pi, \pi)$,

$$P_M[y;\alpha,\tau] \triangleq \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathcal{P}_M[y(t);\tau]\, e^{-j\alpha t}, \qquad (5)$$

which we term the *high-order ambiguity function* (HAF).

Substituting (4) into (5), we obtain $P_M[y;\alpha,\tau] = \rho^{2^{M-1}} e^{j\tilde{\phi}}\, \delta(\alpha - \tilde{\omega})$, where $\delta(\cdot)$ denotes the Kronecker delta function. Therefore, $P_M[y;\alpha,\tau]$ peaks at $\tilde{\omega} = M!\, \tau^{M-1} a_M$, and the highest order polynomial phase coefficient a_M can be obtained from its peak location as

$$a_M = \frac{1}{M!\,\tau^{M-1}} \ \arg\max_{\alpha} \left| P_M[y;\alpha,\tau] \right|. \qquad (6)$$

Next, by multiplying $y(t)$ with $\exp\{-ja_M t^M\}$, we obtain a PPS of order $M-1$. The above procedure is then repeated to yield a_{M-1}. Subsequent iterations yield $a_{M-2}, \ldots, a_1$, cf. [5, Ch. 12]. Finally, $\rho \exp\{ja_0\}$ can be estimated via linear least squares.

In practice, additive noise is present and we receive

$$x(t) = y(t) + v(t) = \rho\, e^{j\sum_{m=0}^{M} a_m t^m} + v(t). \qquad (7)$$

To compute the sample estimate of the HAF, we first substitute $y(t)$ by $x(t)$ in the HIM operator: $\hat{\mathcal{P}}_M[y(t);\tau] \triangleq \mathcal{P}_M[x(t);\tau]$, and then take its normalized DFT,

$$\hat{P}_M[y;\alpha,\tau] \triangleq \frac{1}{T} \sum_{t=0}^{T-1} \hat{\mathcal{P}}_M[y(t);\tau]\, e^{-j\alpha t}. \qquad (8)$$

Asymptotic unbiasedness and consistency of (8) were established in [10] (see also [5, Ch. 12]) when $v(t)$ is zero-mean white Gaussian. Once $\hat{P}_M[y;\alpha,\tau]$ is computed, a_M can be estimated by replacing $P_M[y;\alpha,\tau]$ by $\hat{P}_M[y;\alpha,\tau]$ in (6).

3. MULTI-COMPONENT CHIRP SIGNALS

In Doppler applications and when dealing with multiple moving targets, the returned echo can be modeled as a multi-component PPS,

$$y(t) = \sum_{l=1}^{L} y_l(t) = \sum_{l=1}^{L} \rho_l\, e^{j\sum_{m=0}^{M_l} a_{l,m} t^m} \qquad (9)$$

where each $y_l(t)$ is a constant amplitude PPS of order M_l. When the nonlinear HIM operator $\mathcal{P}_M$ is applied to $y(t)$, many cross-terms will emerge,

$$\mathcal{P}_M[y(t);\tau] = \sum_{l=1}^{L} \mathcal{P}_M[y_l(t);\tau] + \text{cross-terms}. \qquad (10)$$

For an L-component signal with all $M_l = M$, the number of cross-terms is $L^{2^{M-1}} - L$, which gives two cross-terms for $L = 2$, $M = 2$ and 14 for $L = 2$, $M = 3$. Although the L auto-terms become harmonics, these cross-terms remain as PPS. In this section we shall study the $M = 2$ (chirp) case in detail. In particular, we focus on the two-component ($L = 2$) case here because generalization to $L > 2$ components is straightforward. A two-component chirp signal with constant amplitudes is give by

$$y(t) = \rho_1 e^{j(a_{10}+a_{11}t+a_{12}t^2)} + \rho_2 e^{j(a_{20}+a_{21}t+a_{22}t^2)}. \qquad (11)$$

The components are considered as distinct if their respective instantaneous frequencies, $a_{l1} + 2a_{l2}t$, are different.

It is not difficult to show that the 2nd-order instantaneous moment of $y(t)$ in (11) is (assuming $\tau = 1$)

$$\begin{aligned}
&\mathcal{P}_2[y(t);1] \triangleq y(t)y^*(t-1)\\
&= \rho_1^2\, e^{2ja_{12}t}\, e^{j(a_{11}-a_{12})} + \rho_2^2\, e^{2ja_{22}t}\, e^{j(a_{21}-a_{22})}\\
&+ \underbrace{2\rho_1\rho_2\, e^{j\{(a_{12}-a_{22})t^2+(a_{11}-a_{21}+2a_{22})t+(a_{21}-a_{22}+a_{10}-a_{20})\}}}_{\mathcal{T}_1(t)}\\
&+ \underbrace{2\rho_1\rho_2\, e^{j\{(a_{22}-a_{12})t^2+(a_{21}-a_{11}+2a_{12})t+(a_{11}-a_{12}+a_{20}-a_{10})\}}}_{\mathcal{T}_2(t)}.
\end{aligned} \qquad (12)$$

The FS coefficient function of $\mathcal{P}_2[y(t);1]$ is given by

$$\begin{aligned}
&P_2[y;\alpha,1]\\
&= \rho_1^2\, e^{j(a_{11}-a_{12})}\, \delta(\alpha - 2a_{12}) + \rho_2^2\, e^{j(a_{21}-a_{22})}\, \delta(\alpha - 2a_{22})\\
&+ \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathcal{T}_1(t)\, e^{-j\alpha t} + \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathcal{T}_2(t)\, e^{-j\alpha t}.
\end{aligned} \qquad (13)$$

We refer to the first two terms on the r.h.s. of (13) as auto-peaks because their locations yield the highest order polynomial phase coefficients a_{12} and a_{22}. To obtain good estimates of a_{12} and a_{22} based on the locations of the auto peaks, it is thus highly desirable that the last two terms on the r.h.s. of (13), i.e. the FS coefficients of $\mathcal{T}_1(t)$ and $\mathcal{T}_2(t)$, be negligible as compared to the auto-peaks.

3.1. FS Coefficient Function of a Single Chirp

For simplicity, let us rewrite $\mathcal{T}_1(t)$ defined in (12) as

$$\mathcal{T}_1(t) = 2\rho_1\rho_2\, e^{j\gamma_2 t^2}\, e^{j\gamma_1 t}\, e^{j\gamma_0}, \qquad (14)$$

where $\gamma_2 \triangleq a_{12} - a_{22}$, $\gamma_1 \triangleq a_{11} - a_{21} + 2a_{22}$, $\gamma_0 \triangleq a_{21} - a_{22} + a_{10} - a_{20}$. Let us denote the FS coefficient function of $e^{j\gamma_2 t^2}$ by $h(\alpha)$,

$$h(\alpha) \triangleq \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} e^{j\gamma_2 t^2}\, e^{-j\alpha t}, \qquad (15)$$

and express the third term on the r.h.s. of (13) as

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathcal{T}_1(t)\, e^{-j\alpha t} = 2\rho_1\rho_2\, e^{j\gamma_0}\, h(\alpha - \gamma_1). \qquad (16)$$

We would like to compare its magnitude with ρ_l^2, the magnitude of the auto-peaks.

Interestingly, although $\exp(j\gamma_2 t^2)$ is aperiodic in continuous time, it is periodic in discrete time whenever γ_2 is a rational multiple of π. If we write $\gamma_2 = 2\pi N/D$ with $N, D > 0$ co-prime, then D is a period (although it may not be the smallest period). Consequently, $h(\alpha)$ contains spectral lines. Theorem 1 below establishes a bound on $|h(\alpha)|$ and is crucial for assessing the effect of the cross-terms in (13).

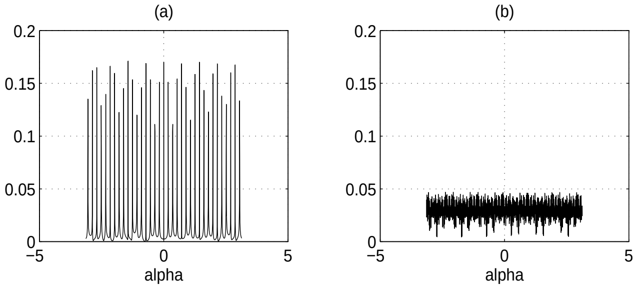

Figure 1. FS coefficient function of $e^{j\gamma_2 t^2}$ for $\gamma_2 = 2\pi \cdot 24/35$ and $\gamma_2 = 0.5$.

Theorem 1 *Suppose that $\gamma_2 = 2\pi N/D$ where $D > 0$ and N are co-prime integers. Then the FS coefficient function $h(\alpha)$ of $e^{j\gamma_2 t^2}$ satisfies*

$$\max_\alpha |h(\alpha)| = \begin{cases} \sqrt{2/D}, & \text{if } D \text{ is even,} \\ \sqrt{1/D}, & \text{if } D \text{ is odd.} \end{cases} \tag{17}$$

Note that the r.h.s of the above does not depend on N.

Our Theorem 3 later on asserts that $h(\alpha)$ of (15) tends to zero *uniformly* in α when γ_2 is an irrational multiple of π. Although any irrational number can be approximated to arbitrary precision by rational numbers, the denominators of those rationals tend to infinity as the precision increases. Theorem 1 then predicts that the corresponding $h(\alpha)$ is negligible in general. The following example illustrates the difference between the two scenarios.

Example 1. Figure 1(a) shows $|h(\alpha)|$ (calculated with $T = 1,024$) as a function of $\alpha \in [-\pi, \pi)$ for $\gamma_2 = 2\pi \cdot 24/35$. We observe 35 spectral lines, and their magnitudes do not exceed $1/\sqrt{D} = 0.169$. In Figure 1(b), we have $\gamma_2 = 0.5$, which cannot be expressed as $2\pi N/D$ for integers D, N. There are no discernible peaks in Figure 1(b) and $|h(\alpha)|$ here is much smaller than that in the previous case.

Because line spectra are produced only when γ_2 is a rational multiple of π, and almost all real numbers are irrational, we conclude that spectral lines appear in $h(\alpha)$ with probability zero and $h(\alpha)$ is very small for large T.

Based on the above observations and together with (16), we infer that the two cross-terms in (13) are negligible and HAF is virtually additive:

$$\begin{aligned} P_2[y;\alpha,1] \quad &\approx \quad \rho_1^2 \, e^{j(a_{11}-a_{12})} \, \delta(\alpha - 2a_{12}) \\ &\quad + \rho_2^2 \, e^{j(a_{21}-a_{22})} \, \delta(\alpha - 2a_{22}). \end{aligned} \tag{18}$$

3.2. Worst Case Scenarios

In [4] Polad and Friedlander proposed a procedure for tracking multi-component PPS parameters. Those of the strongest component are first identified. The component is then removed and the estimation process is continued with the remaining $L-1$ components. Relation $\rho_1/\rho_2 > 2$ was assumed in [4] in order for $\rho_1^2 > 2\rho_1\rho_2$ and so to ensure that the cross-terms are never more than the strongest auto peak. With the help of Theorem 1 (and Theorem 3 in Section 4), however, we can relax the above assumption considerably.

We observe that the contribution from the cross-term in (16) is no more than $2\rho_1\rho_2 \max_\alpha h(\alpha)$, i.e.

$$\lim_{T\to\infty} \left| \frac{1}{T} \sum_{t=0}^{T-1} \mathcal{T}_1(t) \, e^{-j\alpha t} \right| \le 2\rho_1\rho_2 \, \max_\alpha h(\alpha). \tag{19}$$

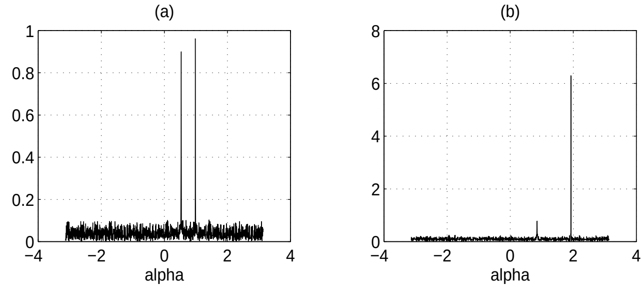

Figure 2. $\hat{P}_2[y;\alpha,1]$ of a two component chirp.

The r.h.s. tends to zero (Theorem 3) when γ_2 is an irrational multiple of π, and is nonzero otherwise. The worst cases are when $\gamma_2 = 2\pi N/D$ with D small, and we shall examine these cases below.

First, we recognize that with $\tau = 1$, the leading chirp coefficients must satisfy $|a_{12}| < \pi/2$ and $|a_{22}| < \pi/2$ in order to satisfy the HAF-based identifiability condition $|M! a_M| < \pi$ ($M = 2$ here). This implies that $|\gamma_2| = |a_{12} - a_{22}| < \pi$, and hence $N/D < 1/2$. Without loss of generality, we assume that $\rho_1 \ge \rho_2$. The worst case scenarios are identified as follows:

(c1) $D = 4$, $N = 1$ and $|a_{12} - a_{22}| = \pi/2$. The r.h.s. of (19) is then $\sqrt{2}\,\rho_1\rho_2$. In order for the cross-term not to exceed the strongest auto peak ρ_1^2, we need $\rho_1/\rho_2 > \sqrt{2}$.

(c2) $D = 3$, $N = 1$ and $|a_{12} - a_{22}| = 2\pi/3$. The r.h.s. of (19) is then $2\rho_1\rho_2/\sqrt{3}$. In order for the cross-term not to exceed the strongest auto peak ρ_1^2, we need $\rho_1/\rho_2 > \sqrt{2}$.

For all other D's, Theorem 1 ensures that the the cross-term in (19) is never more than the strongest auto-peak. Hence, we conclude that if $|a_{12} - a_{22}| \ne \pi/2$ or $2\pi/3$, then the successive estimation algorithm described in [4] can be implemented for *any* $\rho_1/\rho_2 > 1$. Otherwise, one needs to ensure that $\rho_1/\rho_2 > \sqrt{2}$ or $2/\sqrt{3}$. This is a much weaker condition than the one stated in [4].

We further infer from Theorem 1 that if $D \ge 8$ is even and $1 \le \rho_1/\rho_2 < \sqrt{D/8}$, or, if $D \ge 5$ is odd and $1 \le \rho_1/\rho_2 < \sqrt{D}/2$, then the two strongest peaks in $P_2[y;\alpha,1]$ will always be due to the auto-terms, because the r.h.s. of (19) will always be smaller than ρ_2^2 (and hence ρ_1^2). For a generic $\gamma_2 = a_{12} - a_{22}$ to be well approximated by $2\pi N/D$, D would have to be fairly large, and the above condition is then easily met. This implies that in general, $P_2[y;\alpha,1]$ can be regarded as virtually additive, and it is safe to use the locations of the L largest peaks to estimate a_{l2} for $l = 1, 2, \ldots, L$.

Example 2. We generated $T = 1024$ samples of a two-component PPS $y(t)$ given by (11), where each a_{lm} is an i.i.d. uniform random variables in $[0, 1)$. Figures 2(a) and 2(b) show particular realizations of $\hat{P}_2[y;\alpha,1]$ with amplitudes $\rho_1 = \rho_2 = 1$ and $\rho_1 = 2.5$, $\rho_2 = 1$, respectively. We observe two distinct peaks in $\hat{P}_2[y;\alpha,1]$, the locations of which correspond to $2a_{12}$ and $2a_{22}$, illustrating the virtual additivity of $\hat{P}_2[y;\alpha,1]$. This experiment was repeated 100 times, and no cross-terms have ever been observed. Note that although the dynamic range in Figure 2(b) is large due to $\rho_1 \ne \rho_2$, the two strongest peaks nevertheless yield the correct $2a_{12}$ and $2a_{22}$.

Proceeding arguments assume that the leading chirp coefficients are different. The picture changes when $a_{12} = a_{22}$, because then the two auto-peaks merge to one, and $\gamma_2 = 0$ makes both cross-terms $\mathcal{T}_1(t)$ and $\mathcal{T}_2(t)$ behave like harmon-

TABLE I. EXAMPLES OF (20)

b_2	b_3	D	d_D	l.h.s.	r.h.s.
1024708	7286213	11142379	8	0.0813	0.1038
1135718	950919	1247601	8	0.0864	0.2155
1545555	279513	5888885	16	0.0880	0.4085

ics. Under the assumption that the instantaneous frequencies of the different components must be different, we infer that $a_{11} \neq a_{21}$ when $a_{12} = a_{22}$ and $\mathcal{T}_1(t)$ and $\mathcal{T}_2(t)$ generate peaks at $2a_{12} + (a_{11} - a_{21})$ and $2a_{12} - (a_{11} - a_{21})$ in $P_2[y; \alpha, 1]$, which are equidistant from the peak at $2a_{12}$.

4. MULTI-COMPONENT PPS OF ORDER M

For the general L-component constant amplitude PPS model of (9), we assume without loss of generality that the polynomial phase orders satisfy $M_1 \geq M_2 \geq \ldots \geq M_L$. The HAF of order M_1, $P_{M_1}[y; \alpha, \tau]$, exhibits peaks at $M_1! \tau^{M_1-1} a_{l,M_1}$ for all l such that $a_{l,M_1} \neq 0$, but a large number of cross-terms are also present. As in Section 3.1, we shall examine the magnitude of the FS coefficient function of $c(t) = e^{j \sum_{m=2}^{M} a_m t^m}$ in order to make inference about the contribution of those cross-terms.

As with the case of a chirp, $c(t)$ is periodic in discrete time t if and only if all a_m are rational multiples of π. If so the FS coefficient function of $c(t)$ contain spectral lines. Unlike the case of a chirp, for $M > 2$ there is no general formula for the largest magnitude of those spectral lines. Nevertheless, we establish a bound on the largest magnitudes in the following theorem:

Theorem 2 *Consider the polynomial phase signal* $c(t) = e^{j \sum_{m=2}^{M} a_m t^m}$ *and suppose that* $a_m = 2\pi b_m / D$, *where* D, b_1, $\ldots$, b_m *are relatively prime integers,* $D > 0$. *Then*

$$\lim_{T \to \infty} \max_{\alpha} \left| \frac{1}{T} \sum_{t=0}^{T-1} c(t)\ e^{-j\alpha t} \right| \leq \min\left(1,\ c_D / D^{1/M}\right) \quad (20)$$

where $c_D = d_D^{\log_2 M}$ *and* d_D *denotes the number of divisors of* D. *Hence the FS coefficient function of* $c(t)$ *is uniformly bounded by* $\min\left(1,\ c_D / D^{1/M}\right)$.

Remark: It can be shown that the rate of growth of d_D as D increases is approximately logarithmic or less, depending on how many factors D has, see Rosen [7]. Therefore, $\lim_{D \to \infty} d_D / D^{\varepsilon} = 0$ for any $\varepsilon > 0$. As a result, the r.h.s. of (20) tends to zero as $D \to \infty$.

We note here that Theorem 2 provides bounds on the FS coefficient functions of all Mth-order PPS. Since it includes the worst case scenarios, these bounds may not be optimal sometimes. However, the established bound does point out the qualitative dependence of the magnitude of FS coefficient function on $D^{-1/M}$, which tends to zero as $D \to \infty$. The work by Hua [1], Vinogradov [8] and others also indicates that the exponent $-1/M$ of D on the r.h.s. of (20) is optimal and cannot be improved. From (20), we infer that the larger the D and the smaller the d_D, the tighter the bound. For generic $a_m = 2\pi b_m / D$ such will be the case. Table 4. gives numerical examples on the use of (20).

In practice, it is unlikely for an arbitrarily chosen a_m to be a rational multiple of π, and it is even less likely for *all* $\{a_m\}$, $m = 2, \ldots, M$, to be rational multiples of π. The following theorem show the magnitude of the FS coefficient function tends to zero uniformly when the conditions of Theorem 2 are not met.

Theorem 3 *Consider the polynomial phase signal* $c(t) = e^{j \sum_{m=2}^{M} a_m t^m}$ *and suppose that at least one* a_m *is an irrational multiple of* π. *Then*

$$\lim_{T \to \infty} \max_{\alpha} \left| \frac{1}{T} \sum_{t=0}^{T-1} c(t)\ e^{-j\alpha t} \right| = 0. \quad (21)$$

The importance of Theorems 2 and 3 is to guarantee that except in the pathological cases when all a_m are rational multiples of π with a *very small* common denominator, the FS coefficient function of the cross-term $c(t)$ are negligible. Hence in high-order multi-component PPS setting the HAF is virtually additive.

5. CONCLUSIONS

Multi-component AM-FM models describe a large class of nonstationary processes, among which multi-component polynomial phase signals (PPS) form a particularly important subclass. The so-called high-order ambiguity function (HAF) was originally introduced by Peleg and Porat to estimate the parameters of single-component PPS, but has not been widely used for multi-component problems due to the appearance of many cross-terms. In this paper, we have carefully examined the magnitudes of the cross-terms and shown that they are almost always negligible in comparison to the peaks due to the original signal components. Thus HAF can be regarded as virtually additive and be safely applied to multi-component PPS.

Our simulations show that cross-terms rarely cause false peaks in the HAF domain. Problems may arise when the components share the same (highest order) polynomial phase coefficients or when the dynamic range of the component amplitudes is large. Algorithms using HAF for estimation of multi-component PPS is currently under investigation.

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