OFF-LINE DETECTION AND ESTIMATION OF ABRUPT CHANGES CORRUPTED BY MULTIPLICATIVE COLORED GAUSSIAN NOISE

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ABSTRACT

The problem addressed in the paper is the detection of abrupt changes embedded in multiplicative colored Gaussian noise. The multiplicative noise is modeled by an AR process. The Neyman Pearson detector is developped when the abrupt change and noise parameters are known. This detector constitutes a reference to which suboptimal detectors can be compared. In practical applications, the abrupt change and noise parameters have to be estimated. The maximum likelihood estimator for these parameters is then derived. This allows to study the generalized likelihood ratio detector.

1. INTRODUCTION

There is an increasing interest in multiplicative noise models for many signal processing applications such as image processing (speckle). This paper addresses the problems of detection and estimation of abrupt changes corrupted by multiplicative colored Gaussian noise. The signal can be modeled as:

$$x(n) = b(n)\, s(n) \qquad n = 1, ..., N \tag{1}$$

In image processing, $b(n) = m + y(n)$ is the multiplicative speckle noise (with mean m) and $s(n)$ is a line of the image. When intensity images are considered, the statistical properties of $b(n)$ and $s(n)$ can be defined as follows.

The *speckle noise* is usually modeled as a stationary exponentially distributed process. However, in many applications including SAR image processing, the speckle is reduced by incoherently averaging N_i uncorrelated images for large values of N_i [2]. The resulting reduced-speckle intensity images are Gaussian distributed (using the central limit theorem). The simple case of a multiplicative white Gaussian noise has been studied in [9]. However, as it is specified in [2], it is more realistic to model the speckle by a band-limited noise process containing only lower spatial frequencies. In this case, the algorithms developed in [9] cannot be used. Here, $y(n)$ is assumed to be a zero-mean Gaussian stationary AR(p) process with parameters σ^2 and $a = (a_1, ..., a_p)^T$. The multiplicative colored noise $y(n)$ can be modelled by an AR process for the following reasons:

•for any real-valued stationary process $y(n)$ with continuous spectral density $S(f)$, an AR process can be found with a spectral density arbitrarily close to $S(f)$ ([3], p. 130).

•zero-mean Gaussian processes are completely defined by their spectra.

An *ideal abrupt change* can be modelled as a step of amplitude A, located at position n_0 [1],[2]:

$$\begin{array}{ll} s(n) = 1 & n \leq n_0 \\ s(n) = 1 + A & n > n_0 \end{array} \tag{2}$$

Eq. (2) models the edge between two regions with different reflectivities in piecewise constant backgrounds. Abrupt change detection and estimation, corrupted by additive noise, has been studied for long time (see [1] and references therein for an overview). The new contribution here is the development of several detection algorithms for abrupt changes **multiplied by a colored Gaussian noise**.

The first part of the paper studies the optimal Neyman Pearson Detector (NPD). The NPD is optimal in the sense that it minimizes the Probability of Non Detection (PND) for a fixed Probability of False Alarm (PFA). The abrupt change and noise parameters have to be estimated in practical applications. The Maximum Likelihood Estimator (MLE) for these parameters and the Generalized Likelihood Ratio Detector (GLRD) are then studied.

2. NEYMAN PEARSON DETECTOR (NPD)

Under hypothesis H_0, the signal is a stationary zero-mean AR(p) Gaussian sequence $y(n)$ with parameters

σ^2 and $a = [a_1, ..., a_p]^T$ plus a constant mean m

$$x(n) = y(n) + m \tag{3}$$

Under hypothesis H_1, the process $y(n) + m$ is multiplied by an amplitude A abrupt change at time t_0:

$$x(n) = (y(n) + m)\, s(n) \tag{4}$$

Under hypothesis H_i, the likelihood function for the Gaussian vector $X = [x(1), ..., x(N)]^T$ (with mean M_i and covariance matrix Σ_i) denoted $L(X|H_i)$ is defined by:

$$\begin{aligned} \ln L(X|H_i) = & -\tfrac{1}{2}(X - M_i)^T \Sigma_i^{-1}(X - M_i) \\ & -\tfrac{n}{2}\ln(2\pi) - \tfrac{1}{2}\ln|\Sigma_i| \end{aligned} \tag{5}$$

In (5), $|\Sigma_i|$ is the Σ_i matrix determinant, $M_0 = m[1, ..., 1]^T$ and $M_1 = mS$ with $S = [s(1), ..., s(N)]^T$. The inverse covariance matrix of an AR(p) process can be expressed as a function of the model parameters σ^2 and $a = [a_1, ..., a_p]^T$ with the Gohberg-Semencul formula:

$$\Sigma_0^{-1} = \frac{1}{\sigma^2}\left(FF^T - GG^T\right) \tag{6}$$

where F and G are $N \times N$ lower triangular matrices defined for instance in [8]. Under hypothesis H_1, the inverse covariance matrix of the vector X can then be expressed as:

$$\Sigma_1^{-1} = \frac{1}{\sigma^2} D\left(FF^T - GG^T\right) D \tag{7}$$

where $D = diag(1/s(1), ..., 1/s(N))$ is a diagonal $N \times N$ matrix whose elements are $1/s(i)$. The Neyman-Pearson test is defined by:

$$H_0 \text{ rejected if } \quad Q(X) > S(PFA) \tag{8}$$

In (8), $Q(X) = Q_0(X) - Q_1(X)$ where $Q_0(X)$ and $Q_1(X)$ are the two positive definite quadratic forms

$$\begin{aligned} Q_0(X) &= (X - M_0)^T \Sigma_0^{-1}(X - M_0) \\ Q_1(X) &= (X./S - M_0)^T \Sigma_0^{-1}(X./S - M_0) \end{aligned}$$

The operator $(./)$ denotes the element by element vector division. Note that the computation of $Q_0(X)$ and $Q_1(X)$ can lead to a high computational cost since F and G are $N \times N$ matrices. To solve this problem, $Q_0(X)$ and $Q_1(X)$ can be expressed as quadratic forms of the AR parameters. This leads to a simple computation of the test statistics $Q(X)$ using $p \times p$ matrices [5]. The quadratic form $Q(X)$ is indefinite in general. Relatively little attention has been devoted to the problem of obtaining the distribution of indefinite quadratic forms of Gaussian vectors. Laguerre or Maclaurin series expansions, or expansions as mixtures of noncentral χ^2 distributions have been derived. However, these expansions are difficult to study [6]. Instead, the distribution of $Q(X)$ can be approximated, leading to a simple test statistic. For example, Fig.'s 1.a) and b) show that the Probability Density Function (PDF) of $Q(X)$ can be approximated by the Gaussian PDF under hypotheses H_0 and H_1.

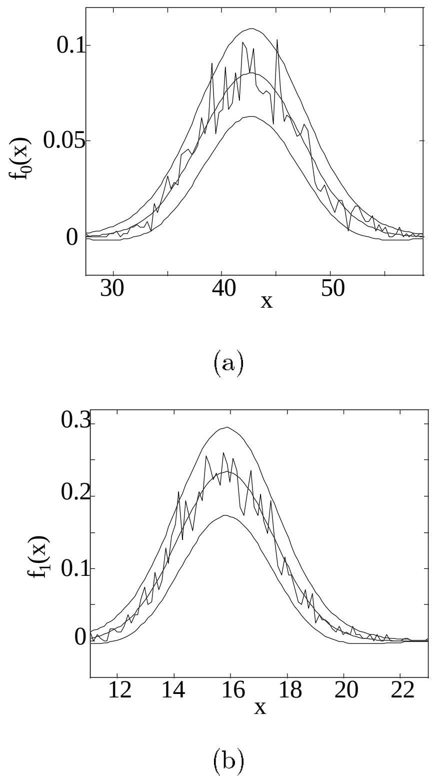

Fig 1. Histograms and PDF of $Q(X)$ with 95% confidence intervals a) under H_0 b) under H_1

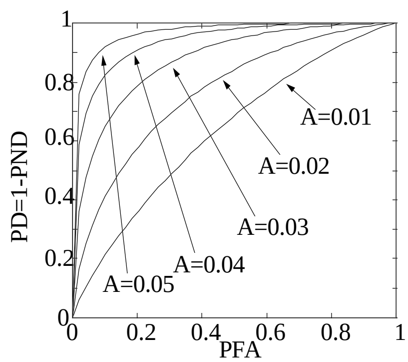

Fig. 2. ROC Curves for the NPD

The approximated ROC curves are shown in Fig. 2 as a function of the abrupt change amplitude A. As it can be seen, for $A > 0.05$, the test shows very good performance. The noise and abrupt change parameters are unknown in practical applications. Thus, the parameters have to be estimated. The next part of the paper derives the Maximum Likelihood Estimator (MLE) for $\theta = \left(m, \sigma^2, a, A, n_0\right)^T$.

3. MAXIMUM LIKELIHOOD ESTIMATOR

The maximum likelihood principle provides a method to estimate a parameter vector θ from a finite length data record $X = [x(1), ..., x(N)]^T$. Under hypothesis H_0, $X - M_0$ is a Gaussian AR(p) process whose parameters can be estimated with the conventional autocorrelation or covariance methods [7]. This section focuses on estimating the noise and abrupt change parameters under hypothesis H_1. The noise and abrupt change parameters are $\left(m, \sigma^2, a\right)$ and (A, n_0) such that $\theta = \left(m, \sigma^2, a, A, n_0\right)^T$ with $a = [a(1), ..., a(p)]^T$. The exact maximization of the likelihood function of the Gaussian vector X produces a set of highly non-linear equations, even in the pure AR case ($A = 0$) [8]. The maximization of the likelihood function can be approximated by maximizing the conditional likelihood function for large data records ([7], p. 186):

$$L(x(p+1), ..., x(N) \mid x(1), ..., x(p); \theta) \tag{9}$$

The driving AR(p) process $u(n)$ is assumed an i.i.d. sequence with zero mean and variance σ^2. The Jacobian matrix determinant of the transformation from $\widetilde{U} = [u(p+1), ..., u(N)]^T$ to $\widetilde{X} = [x(p+1), ..., x(N)]^T$ is $|J| = \left[\prod_{i=p+1}^{N} s(i)\right]^{-1}$. Consequently, the PDF for $\widetilde{X}$ conditioned on the p first values $x(1), ..., x(p)$ can be determined:

$$\begin{aligned} L\left(\widetilde{X}\right) &= L\left(\widetilde{X} \mid x(1), ..., x(p); \theta\right) \\ &= \frac{|J|}{(2\pi\sigma^2)^{(N-p)/2}} \exp f\left(x; m, \sigma^2, a, s\right) \end{aligned} \tag{10}$$

with

$$\begin{aligned} f\left(x; m, \sigma^2, a, s\right) = -\frac{1}{2\sigma^2} \sum_{i=p+1}^{N} \Bigg(\frac{x(i)}{s(i)} + \\ \sum_{k=1}^{p} a(k) \frac{x(i-k)}{s(i-k)} - m\left[1 + \sum_{k=1}^{p} a(k)\right]\Bigg)^2 \end{aligned} \tag{11}$$

Setting the partial derivatives of $\ln L\left(\widetilde{X}\right)$ with respect to m and σ^2 to zero and replacing m and σ^2 in $\ln L\left(\widetilde{X}\right)$ by their estimates, the criterion J_1 has to be maximized with respect to a, A and n_0 where

$$J_1\left(\widetilde{X}; a, A, n_0\right) = -\sum_{i=n_0+1}^{N} \ln s(i) - \frac{N-p}{2} \ln Q(a) \tag{12}$$

J_1 is maximized over a by minimizing $Q(a)$. Note that $Q(a)$ is a quadratic form in a. As a result, its differentiation yields a global minimum (which may not be unique) defined by a matrix equation denoted as $W\widehat{a}_{ML} = -w$. $\widehat{a}_{ML}$ is then substituted in (12). The maximization of $L\left(\widetilde{X} \mid x(1), ..., x(p); \theta\right)$ over the whole parameter vector θ is equivalent to the maximization of $J_2\left(\widetilde{X}; A, n_0\right) = J_1\left(\widetilde{X}; \widehat{a}_{MV}, A, n_0\right)$ with respect to $(A, n_0)^T$ only. Thus, the MLE for the parameter vector $(A, n_0)^T$ is:

$$\widehat{n_0} = \arg \max_{1 \le k \le N} \left\{ \sup_A J_2\left(\widetilde{X}; A, k\right) \right\} \tag{13}$$

$$\widehat{A} = \arg \sup_A J_2\left(\widetilde{X}; A, \widehat{n_0}\right) \tag{14}$$

This case is significantly more complicated than the white Gaussian multiplicative noise case [9]. The differentiation of $J_2\left(\widetilde{X}; A, n_0\right)$ with respect to A yields a set of non-linear equations which do not lead to an analytical closed form expression of $\widehat{A}$. Consequently, a numerical method has to be used for the estimation of $\sup_A J_2\left(\widetilde{X}; A, k\right)$. This paper proposes to use the conventional iterative quasi-Newton BFGS algorithm (available in the Matlab optimization toolbox). Partial derivatives are computed using a numerical differentiation method via finite differences (although they can be analytically derived with higher computational cost). In general, the cost function $J_2\left(\widetilde{X}; A, k\right)$ has several local maxima. Thus, the optimization procedure has to be initialized sufficiently close to the global maximum.

Abrupt change instant initialization

When the multiplicative noise is non-zero mean ($m \neq 0$), there is a simultaneous mean value and variance jump after the abrupt change instant. The off-line estimation procedure described in ([1], p. 66) for abrupt mean changes then can be used for the initialization of $\widehat{n_0}$.

A mean value jump occurs in the signal $x^2(n)$, when the multiplicative noise is zero-mean ($m = 0$). This jump can be used for the initialization of $\widehat{n_0}$.

Abrupt change amplitude initialization

After the abrupt change instant has been estimated, the amplitude A can be estimated by the ratio of the means before and after the change instant.

The Mean Square Errors (MSE) of $\widehat{A}_{ML}$ and $\widehat{n_0}_{ML}$ computed with $N_r = 500$ Monte-Carlo runs are depicted in Fig. 3, for $m = 1, \sigma^2 = 1, A = 0.5, n_0 = \frac{N}{2}$, versus the number of samples. The comparison with the true parameters shows the ML algorithm efficiency.

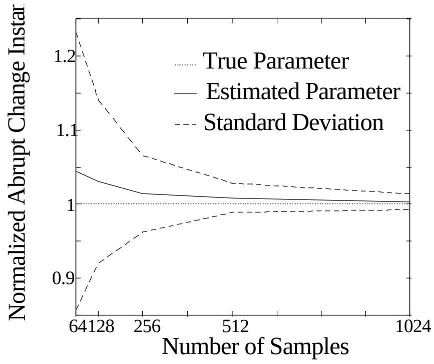

Fig. 3.a) Mean Square Error of $\widehat{n_0}$.

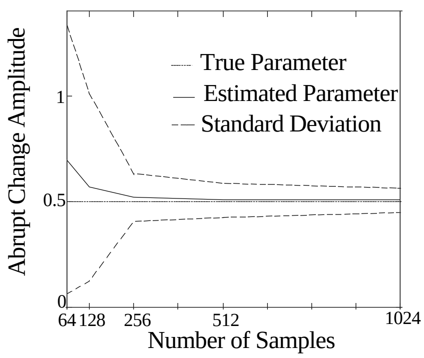

Fig. 3.b) Mean Square Error of $\widehat{A}$.

4. GENERALIZED LIKELIHOOD RATIO DETECTOR (GLRD)

The Generalized Likelihood Ratio Detector (GLRD) estimates the unknown parameters under hypotheses H_0 and H_1 using the maximum likelihood procedure and uses these estimates in the Neyman-Pearson test defined in (8). The GLR test for our problem is:

$$H_0 \text{ rejected if } \quad \frac{L\left(X \middle| \widehat{\theta}_{1ML}\right)}{L\left(X \middle| \widehat{\theta}_{0ML}\right)} > k(PFA) \quad (15)$$

where $\widehat{\theta}_{iML}$ denotes the MLE of θ under hypothesis H_i. The GLR test performances are depicted in Fig. 4 as a function of the abrupt change amplitude. No optimality property can be obtained for the abrupt change detection problem when the abrupt change instant is unknown, even if the parameters before and after change are known [4]. However, Fig. 4 shows that the GLRD has a good performance for an abrupt change amplitude $A \geq 0.2$.

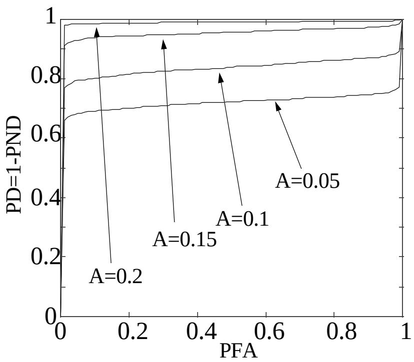

Fig. 4. ROC Curves for the GLRD

5. CONCLUSION

The optimal Neyman-Pearson Detector (NPD) was studied for the abrupt change detection, in presence of a multiplicative colored Gaussian noise. The NPD provides a reference to which suboptimal detectors can be compared. However, the NPD requires knowledge of the abrupt change and noise parameters. The abrupt change and noise parameters are unknown in practical applications and have to be estimated. The Maximum Likelihood Estimator (MLE) for these parameters was then derived. The MLE procedure combined with the NPD yield the Generalized Likelihood Ratio Detector (GLRD). The GLRD performance was studied as a function of the abrupt change amplitude.

6. REFERENCES

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