STATIONARITY ASSESSMENT WITH TIME-VARYING AUTOREGRESSIVE MODELING

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ABSTRACT

A new method for assessing the stationarity of a signal is addressed. The proposed technique is based on the application of time-varying autoregressive models, in which coefficient variations are decomposed upon a set of deterministic basis functions. Stationarity is evaluated by selecting the optimal number of basis functions with a generalized version of Minimum Description Length criterion. Results are then validated with hypothesis testing on the model coefficients. Several simulation results are presented. First, application to synthetic signals confirms the basic assumptions and highlights the main features of the method. Second, relevant conclusions are derived for the study of the stationarity of heart rate time series before the onset of ventricular tachyarrhythmias.

1. INTRODUCTION

Numerous signals such as speech or biomedical ones have their statistical and spectral properties changing with time. Generally, these characteristics are referred to as a *nonstationary* behaviour. Since many signal processing analysis techniques (parametric and nonparametric methods, such as autoregressive (AR) modeling or discrete Fourier transform (DFT)) rely on an assumption of stationarity, assessment of this feature is of great importance.

Different approaches have been described in the literature [1, 2] but their implementation is not always straightforward and the results are sometimes hard to interpret. Moreover, tests based on the computation of statistical parameters (such as in [2]) often require strict conditions (number of samples, number of division intervals). The main goal of this paper is to consider the problem of stationarity assessment with a new objective criterion.

The method introduced here is directly derived from a parametric approach. The classical AR modeling scheme is transformed into a time-varying autoregressive (TVAR) model by decomposing the time evolution of the coefficients as a linear combination of basis functions. Then, the well known *Minimum Description Length* (MDL) criterion for AR order selection is generalized in order to determine the optimal number of these basis functions. Stationarity is finally assessed by considering the optimal basis order and by performing a hypothesis test on the TVAR coefficients.

2. TIME-VARYING AUTOREGRESSIVE MODEL

This way of expressing time dependence in the AR coefficients seems to have been first introduced by Rao in [3]. Further studies have been reported by Liporace [4], Grenier [5], and Hall *et al.* [6].

A scalar discrete-time stochastic process can be expressed with the following classical AR model:

$$x(n) = \sum_{i=1}^{p} a_i x(n-i) + e(n) \tag{1}$$

where $e(n)$ is the driving white noise, and a_i are the p-order model coefficients. However, when $x(n)$ cannot be considered as a stationary process, this approach is no longer valid since there is no time dependence in the AR coefficients. A solution is then to make the assumption that the coefficient variations can be approximated by a linear combination of a finite number of known deterministic functions $u_k(n)$ (called a *basis*):

$$a_i(n) = \sum_{k=0}^{q} a_{ik} u_k(n) \tag{2}$$

Therefore, the general TVAR model can be formulated as:

$$\begin{aligned} x(n) &= \sum_{i=1}^{p} a_i(n) x(n-i) + e(n) \\ &= \sum_{i=1}^{p} \Big(\sum_{k=0}^{q} a_{ik} u_k(n)\Big) x(n-i) + e(n) \end{aligned} \tag{3}$$

Estimation of the coefficients a_{ik} is achieved by minimizing the total mean square error. This leads to the computation of a generalized correlation function given by:

$$c_{kl}(i,j) = \sum_{n} u_k(n) u_l(n) x(n-i) x(n-j) \tag{4}$$

Determination of the coefficients a_{ik} themselves is then performed by solving the following set of normal equations:

$$\sum_{i=1}^{p} \sum_{k=0}^{q} a_{ik} c_{kl}(i,j) = c_{0l}(0,j) \tag{5}$$
$$1 \le j \le p \quad \text{and} \quad 0 \le l \le q$$

Hall *et al.* [6] have discussed in more detail the computational aspects of this model.

3. PARAMETER SELECTION

Unlike the classical AR modeling scheme, the parameter choice depends on three degrees of freedom, namely the AR order p, the basis order q, and the set of basis functions $u_k(n)$.

3.1. Choice of the Basis Functions

According to Equation 2, no particular constraint is imposed on the basis $u_k(n)$. Therefore, one will be able to track only variations which are approximable by this set of functions. Many different solutions have been proposed, among which a Fourier basis [6], a wavelet basis [7], or a set of prolate spheroidal functions [8]. None of these solutions seems to be definitive, partly because the choice of $u_k(n)$ needs some *a priori* knowledge upon the time variations present in $x(n)$. On the other hand, bases such as prolate spheroidal functions are very complex and hard to generate. We suggest here to use classical polynomial functions (namely Chebychev, Hermite and Legendre polynomials) because they can approximate a wide range of variations and their implementation is straightforward.

3.2. Order Selection

The usual way of choosing the order of a model is to make use of a selection criterion. In this work, Rissanen's Minimum Description Length (MDL) criterion [9] is generalized to the TVAR model. The main idea, based on information theory, starts with the notion that the length required to encode a set of observations depends on the model that is assumed to have generated these data. Accordingly, Rissanen selects the model that minimizes the code length of the observed data. In order to take into account the modifications introduced by the TVAR scheme, the MDL criterion can be adapted according to the following equation:

$$\mathrm{MDL}(p,q) = N \ln(\sigma^2_{p,q}) + p(q+1)\ln(N) \qquad (6)$$

where N is the signal length, and $\sigma^2_{p,q}$ is the residual error variance. Minimization of (6) by an exhaustive search on a set of $(p;q)$ values gives the optimal orders $(p_{opt};q_{opt})$.

4. MODEL VALIDATION

We also propose to compute the statistical significance of the TVAR coefficients in order to validate the optimal model. For this, we first make some assumptions about the distribution of the estimated coefficients. This can be achieved by deriving the expression of the maximum likelihood estimator $L(\mathbf{a})$, where $\mathbf{a}$ is the vector of estimated coefficients:

$$\ln L(\mathbf{a}) = -\left[\frac{\mathbf{N-p+1}}{\mathbf{2}}\ln(\mathbf{2}\pi\sigma^{\mathbf{2}})\right] - \frac{1}{2\sigma^2}\sum_{n=p}^{N}\left[x(n) - \sum_{i=1}^{p}\left(\sum_{k=0}^{q} a_{ik}u_k(n)\right)x(n-i)\right]^2 \qquad (7)$$

As is well known, under certain regularity conditions, these estimators are asymptotically multivariate normally distributed with covariance matrix equal to the inverse of the Hessian or *information matrix*:

$$\mathbf{F} = \frac{\partial^2 \log \mathbf{L(a)}}{\partial \mathbf{a} \partial \mathbf{a^T}} \qquad (8)$$

The diagonal elements of the inverse Hessian give thus approximations of the variances $\sigma^2_{a_{ik}}$ of the model coefficients [10].

Therefore, under the null hypothesis that a coefficient is zero, the ratio of the coefficient divided by its standard deviation is approximately distributed as a standard normal variable. We can thus compute the probability:

$$prob(|a_{ik}| \leq d\sigma_{a_{ik}}) = prob(|a_{ik}/\sigma_{a_{ik}}| \leq d)$$

According to the value of d, the hypothesis is accepted with a significance level α given by the normal distribution $N(\mu = 0, \sigma = 1)$. If one takes $\alpha = 0.95$, a coefficient will be considered statistically significant if the ratio $|a_{ik}/\sigma_{a_{ik}}|$ is greater than 1.96. This can be repeated for each coefficient so as to obtain a global *significance rate* for the model.

5. STATIONARITY ASSESSMENT

Finally, the following algorithm is presented to evaluate the stationarity of a signal.

1. TVAR models are applied for the range $p = p_{min}...p_{max}$ and $q = 0...q_{max}$.
2. Generalized MDL criterion is computed for each couple $(p;q)$.
3. Optimal orders $(p_{opt};q_{opt})$ are those for which the MDL value is the lowest.
4. The number of statistically significant coefficients is derived by performing the above described hypothesis test. The global significance rate is derived.

To assess stationarity, we look at the basis order q_{opt} given by the generalized MDL criterion. If this value is high, it means that a large number of basis functions is required to accurately model the signal. In other words, the statistical properties of the signal are strongly changing with time, suggesting in this way a nonstationary behaviour. On the other hand, if $q_{opt} = 0$, the optimal basis contains only a constant function. Since in this case a TVAR model is equivalent to a standard AR model, we can expect that the signal structure is very close to stationarity. It is also possible that the TVAR model cannot fit the signal under study, for instance with a highly nonlinear signal. So, the low significance level of the TVAR coefficients will indicate that the model is not able to track the signal variations. Hypothesis testing is thus a way of assessing the confidence we can have in the results.

Moreover, it is sometimes useful to determine if a smaller part of a signal is stationary. For this purpose, we suggest to apply the algorithm to subsections of the signal in order to evaluate a possible *horizon of stationarity*, i.e. the length of the greatest signal section which can be considered as stationary.

6. SIMULATION RESULTS

This part of the paper addresses three simulation results in order to validate the main assumptions of the proposed algorithm.

$a_{10} = -1$	$a_{11} = 0.9$	$a_{12} = -0.9$
$a_{20} = -1$	$a_{21} = -0.8$	$a_{22} = -0.8$

Table 1. TVAR coefficients for the first simulation.

	a_{10}	a_{11}	a_{12}	a_{20}	a_{21}	a_{22}
a_{ik}	-1	0.9	-0.9	-1	-0.8	-0.8
$\bar{a_{ik}}$	-0.93	0.84	-0.85	-1.00	-0.78	-0.81
${\sigma_{a_{ik}}}^2$	0.27	0.09	0.37	0.28	0.10	0.39
${\sigma_{a_{ik}}}^2$	0.26	0.11	0.37	0.29	0.12	0.41

Table 2. First row: actual coefficient values. Second row: means of the estimated coefficients for 100 realizations. Third row: means of the estimated variances for each coefficient (computed according to Equation 8). Fourth row: variances of the estimated coefficients for 100 realizations.

6.1. Computation of Coefficient Variances

The first example is a signal of length $N = 500$ built with a TVAR model ($p = 2; q = 2$) and a Chebychev basis. The TVAR coefficients are given in Table 1.

100 realizations of the process were synthetized with unit variance white noise excitation. Simulation results are presented in Table 2. It clearly appears that the variances computed with the inverse Hessian are reliable estimators of the actual coefficient variances.

The algorithm was then applied for p and q ranging from 1 to 5 and with a Chebychev basis in order to retrieve the original values $p_{opt} = 2$ and $q_{opt} = 2$. As expected, all coefficients were found significant.

6.2. Determination of a Stationarity Horizon

The second example is an AR process ($p = 2; q = 0$) of length 500 whose coefficients are changing at the middle of the signal according to the values given in Table 3.

The stationarity test was applied for p ranging from 1 to 5 and q ranging from 0 to 10, with a Legendre basis. The first experiment was carried out on the first 250 samples of the signal, corresponding to a classical AR model. For this part, results were: $p_{opt} = 2$, $q_{opt} = 0$ and a signification rate of 1, confirming the actual values. The second one was on the whole signal and gave the following results: $p_{opt} = 2$, $q_{opt} = 5$ and a significance rate of 0.75. The large value q_{opt} is due to the abrupt change at the middle of the signal. Accordingly, a 250-sample stationarity horizon can be associated with this signal. However, it is clear that in this case, a basis built with discontinuous functions (for instance with Walsh functions) would yield a better modeling.

$1 \leq n \leq 250$	$251 \leq n \leq 500$
$a_{10} = -1$	$a_{10} = 1$
$a_{20} = -0.2$	$a_{20} = -0.4$

Table 3. TVAR coefficients for the second simulation.

6.3. Simulation with a Nonstationary Signal

Finally, we consider the following *heteroscedastic* process [11] generated with a signal-dependent gain and represented in Figure 1:

$$x(n) = 0.5x(n-1) + |x(n-1)|^{0.2}e(n) \qquad (9)$$

This kind of signal displays stationary sections alternating with burst-like nonstationary ones. The number of bursts is conditioned upon the gain multiplying the excitation $e(n)$. Iterations were carried out for p covering the range [5,20] and q covering the range [0,10]. We got maximum values for the orders: $p_{opt} = 20$, $q_{opt} = 10$ and a signification rate of 0.98. The strong nonstationarities present in the signal are related to the high order values.

7. APPLICATION TO BIOMEDICAL DATA

The method described here was initially developed as part of a study dealing with heart rate (HR) signals retrieved from electrocardiogram (ECG) recordings. An example of such a signal for a healthy person is presented in Figure 2. HR variability analysis is a well known technique to investigate interactions between the autonomic nervous system and the cardiovascular system. Classical methods used up to now (parametric and nonparametric power spectrum estimation) rely on an assumption of stationarity. This hypothesis is not obvious since long-term HR recordings have shown strong circadian variations suggesting a nonstationary behaviour. However, current results are still contradictory and there is a need for a better estimation of changes occuring in the HR dynamics.

The goal of this study was to assess the stationarity of HR signals coming from two populations. The first one was a control group of healthy subjects, whereas the second one consisted of patients suffering from ventricular tachyarrhythmias (VTA, tachycardias and fibrillations). In this case, data were retrieved from defibrillators and contained the last 1024 beats before the onset of the VTA.

The following parameters were used for experiments: Legendre basis, TVAR order p ranging from 5 to 30 and basis order q covering the range 0 to 10. Each signal was divided into 6 sections of increasing duration (with a step size of 100 s), in order to determine a possible stationarity horizon. The test was applied successively on each section. It was noted that the optimal basis order q_{opt} was nearly always equal to zero for the two groups and for all sections. The strong time consistency of the orders suggested that up to a 10-minute duration, the HR signal structure is stable and accordingly very close to stationarity. Moreover, no significant difference was observed between the control group and the patients suffering from VTA, invalidating the assumption of a modification of the HR dynamics within 10 minutes preceding a VTA. This result is important because it suggests to consider a wider range of time before the onset of an arrhythmic event to track possible changes in the signal behaviour. Concerning the TVAR signification rates, a lower level was observed for healthy subjects, suggesting an increasing trend in the linearity of the signals, from normal persons to patients suffering from VTA. This observation should motivate further investigations with nonlinear tools.

8. CONCLUSION

A new method for assessing the stationarity of time series was presented in this paper. Simulations showed that the combination of a time-varying autoregressive model and a generalized version of MDL criterion was able to evaluate the stationarity of the signal under investigation. The fitting of the TVAR model to the signal was taken into account by hypothesis testing. Several examples were presented in the paper to highlight the main features of the method. Application to biomedical data was also considered and important conclusions concerning heart rate dynamics were established with this new test.

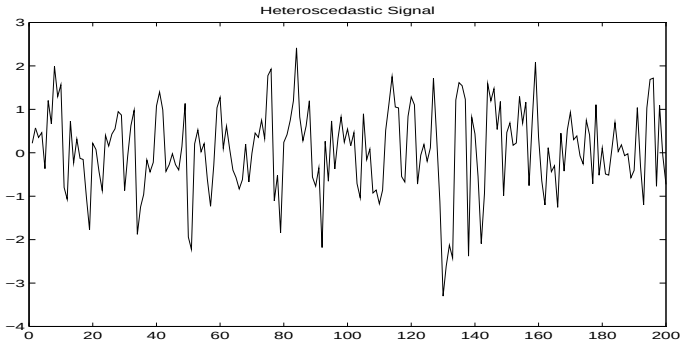

Figure 1. Heteroscedastic signal

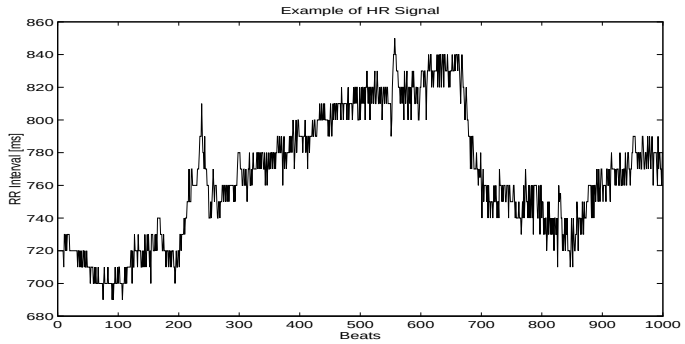

Figure 2. Example of heart rate signal

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