

NEARFIELD BEAMFORMING USING NEARFIELD/FARFIELD RECIPROCITY

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ABSTRACT

We establish the asymptotic equivalence, up to complex conjugation, of two problems: (i) determining the nearfield performance of a farfield beampattern specification, and (ii) determining the equivalent farfield beampattern corresponding to a nearfield beampattern specification. Using this reciprocity relationship we develop a computationally simple procedure to design a beamforming array to achieve a desired nearfield beampattern response. The superiority of this approach to existing methods, both in ease of design implementation and performance obtained, is illustrated by a design example.

1. INTRODUCTION

The majority of array processing literature deals with the case in which the source is assumed to be in the farfield of the array, and hence the received wavefront from a single point source is planar. This assumption significantly simplifies the beamformer design problem. The common rule-of-thumb is that farfield operation can be assumed for sources at a distance of $r = 2L^2/\lambda$, where r is the radial distance from an arbitrary array origin, L is the largest array dimension, and λ is the operating wavelength [1]. However, in many practical situations the source is well within this distance, and using the farfield assumption to design the beamformer results in severe degradation in the beampattern. Despite this, nearfield beamforming is a problem which has been largely ignored in the signal processing literature.

One common method of nearfield beamforming is *nearfield compensation* (e.g., [2]) in which a delay correction is used on each sensor to account for the nearfield wavefronts, which tend to be spherical. Designs based on nearfield compensation tend only to achieve the desired nearfield beampattern over a limited range of angles, because they focus the array to a single point in three dimensional space.

In [3] it was shown how, based on the uniqueness of the solution to the wave equation, any nearfield beampattern

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specification can be transformed to an equivalent farfield specification. However, this nearfield-farfield transformation is computationally involved. In this paper we develop a computationally simple procedure in which *farfield* design techniques may be used directly to design a beamformer to achieve a desired *nearfield* beampattern specification.

2. PROBLEM FORMULATION

2.1. Wave Equation Formulation

At the physical level beamforming is characterised by the wave equation. In the engineering literature this detail of modelling is usually unnecessary as much simpler formulations can be made. However, in our work it is essential to re-explore the well studied wave equation to reveal a non-trivial asymptotic relationship which is central to our novel design procedure.

Let r denote radial distance, ϕ and θ the azimuth and elevation angles. Then a general valid beampattern, $B \equiv B(r, \theta, \phi, t)$, will satisfy the wave equation expressed in spherical coordinates [4]

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial B}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial B}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 B}{\partial \phi^2} = \frac{1}{c^2} \frac{\partial^2 B}{\partial t^2}. \quad (1)$$

In formulating a solution to this equation the time variation can be omitted as it clearly "separates" (as a variable) from the space variables.

The solution is classical [4] and general solutions, $B(r, \theta, \phi)$, can be written as linear combinations of modes of the form

$$r^{-\frac{1}{2}} \begin{matrix} J_{n+\frac{1}{2}}(kr) \\ Y_{n+\frac{1}{2}}(kr) \end{matrix} P_n^m(\cos \theta) \begin{matrix} \cos m\phi \\ \sin m\phi \end{matrix} \quad (2)$$

where integers m and n index the modes,

$$k \triangleq \frac{2\pi f}{c} = \frac{2\pi}{\lambda} \quad (3)$$

is the wave number which can be expressed in terms of the propagation speed c and the frequency f , or the wavelength λ ; $P_n^m(\cdot)$ is the associated Legendre function; $J_{n+\frac{1}{2}}(\cdot)$ is the half odd integer order Bessel Function of the first kind and

$Y_{n+\frac{1}{2}}(\cdot)$ is the half odd integer order Neumann Function (or Bessel Function of the second kind) [4]. Note that in (2) either the upper or lower function in the r and ϕ portions can be taken leading to four possibilities. Finally, when $m = 0$ the associated Legendre function is called the Legendre function $P_n(\cdot)$ [4].

2.2. Beampattern Formulation

In this work we use complex combinations of the classical modes (2) which leads to a more suitable engineering reformulation. We have, equivalent to (2),

$$r^{-\frac{1}{2}} \begin{matrix} H_{n+\frac{1}{2}}^{(1)}(kr) \\ H_{n+\frac{1}{2}}^{(2)}(kr) \end{matrix} P_n^m(\cos\theta) \begin{matrix} e^{+jm\phi} \\ e^{-jm\phi} \end{matrix} \quad (4)$$

where the radial dependency now comes through the half odd integer order Hankel Functions of the first kind and second kind, respectively

$$H_{n+\frac{1}{2}}^{(1)}(\cdot) \triangleq J_{n+\frac{1}{2}}(\cdot) + jY_{n+\frac{1}{2}}(\cdot) \quad (5a)$$

$$H_{n+\frac{1}{2}}^{(2)}(\cdot) \triangleq J_{n+\frac{1}{2}}(\cdot) - jY_{n+\frac{1}{2}}(\cdot) \quad (5b)$$

which form a complex conjugate pair.

An observation regarding the modes in (4) is their magnitude decays to zero magnitude as r approaches infinity and, hence, every wave equation solution $B(r, \theta, \phi)$ has this property. It is desirable to define a beampattern as a function of angles only such that the magnitude remains finite at infinity. Hence, we define the beampattern through

$$b_r(\theta, \phi) \triangleq r B(r, \theta, \phi). \quad (6)$$

Therefore, we write the general solution to (1) in the *beampattern* form (synthesis equation)

$$\begin{aligned} b_r(\theta, \phi) = & \sum_{n=0}^{\infty} \alpha_n r^{\frac{1}{2}} H_{n+\frac{1}{2}}^{(1)}(kr) P_n(\cos\theta) \\ & + \sum_{n=1}^{\infty} \sum_{m=1}^n r^{\frac{1}{2}} H_{n+\frac{1}{2}}^{(1)}(kr) P_n^m(\cos\theta) (\beta_n^m e^{jm\phi} + \gamma_n^m e^{-jm\phi}) \end{aligned} \quad (7)$$

where the complex constants α_n , β_n^m and γ_n^m are Fourier-like coefficients. Note that we have restricted the solution of the wave equation solution (7) to use only the Hankel Functions of the first kind since we wish to exclude standing wave solutions and consider the wavefronts moving towards the origin.

With regard to defining some analysis equations complementing the synthesis equation (7), we have:

$$\alpha_n = \frac{\zeta_n \mathcal{A}_n(b)}{r^{\frac{1}{2}} H_{n+\frac{1}{2}}^{(1)}(kr)} \quad (8a)$$

$$\beta_n^m = \frac{\zeta_n^m \mathcal{B}_{mn}(b)}{r^{\frac{1}{2}} H_{n+\frac{1}{2}}^{(1)}(kr)} \quad (8b)$$

$$\gamma_n^m = \frac{\zeta_n^m \mathcal{C}_{mn}(b)}{r^{\frac{1}{2}} H_{n+\frac{1}{2}}^{(1)}(kr)}, \quad (8c)$$

where r is the radius corresponding to the $b \equiv b_r(\theta, \phi)$ specification,

$$\zeta_n^m \triangleq \left(\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!} \right)^{0.5} \quad (9)$$

with $\zeta_n \triangleq \zeta_n^0$, and

$$\mathcal{A}_n(b) \triangleq \zeta_n \int_0^{2\pi} \int_0^\pi b(\theta, \phi) P_n(\cos\theta) \sin\theta \, d\theta \, d\phi \quad (10a)$$

$$\mathcal{B}_{mn}(b) \triangleq \zeta_n^m \int_0^{2\pi} \int_0^\pi b(\theta, \phi) P_n^m(\cos\theta) \sin\theta \, e^{-jm\phi} \, d\theta \, d\phi \quad (10b)$$

$$\mathcal{C}_{mn}(b) \triangleq \zeta_n^m \int_0^{2\pi} \int_0^\pi b(\theta, \phi) P_n^m(\cos\theta) \sin\theta \, e^{jm\phi} \, d\theta \, d\phi. \quad (10c)$$

Based on this development we can make some observations:

1. Since the complex coefficients, (8a)–(8c) in the expansion (7) completely characterise the beampattern at all distances, the beampattern response can be reconstructed at arbitrary points in space.
2. The $B(r, \theta, \phi)$ expansion analogous to (7) differs only in the factors $r^{-\frac{1}{2}}$ replacing $r^{\frac{1}{2}}$ in (7). The coefficients α_n , β_n^m , γ_n^m , (8a)–(8c), apply in either case.
3. The coefficients (10a)–(10c) represent modal amplitudes and depend only on the shape of the beampattern and not on the radius of the sphere on which the beampattern is given, e.g., the computation is identical whether the beampattern is nearfield or farfield.
4. There is a significant computational burden in accurately evaluating the coefficients (8a)–(8c) because of the multi-dimensional integration necessary from (10a)–(10c). Overcoming this complication is the major motivation for the development of our novel scheme.

3. RADIAL TRANSFORMATIONS

3.1. Key Relationship

The objective is to relate a beampattern specification given on a sphere at one radius, say r_1 from the origin, to a beampattern specification at a second radius, say r_2 from the origin. This is achieved by beampattern analysis at r_1 (through (8a)–(8c)) and resynthesis at r_2 (through (7)). The key technical observation we make is that this problem is essentially identical to the problem of beampattern analysis at r_2 and resynthesis at r_1 (for a different solution to the wave equation), up to complex conjugation and an error term which is typically small for problems of interest.

Proposition 1 *Let $H_{n+\frac{1}{2}}^{(1)}(\cdot)$ and $H_{n+\frac{1}{2}}^{(2)}(\cdot)$ denote the half odd integer order Hankel functions of the first and second*

kinds, respectively, where n is the modal index, λ is the wavelength and $k = 2\pi/\lambda$ the wave number. Then

$$\frac{r_1^{\frac{1}{2}} H_{n+\frac{1}{2}}^{(1)}(kr_1)}{r_2^{\frac{1}{2}} H_{n+\frac{1}{2}}^{(1)}(kr_2)} = \frac{r_2^{\frac{1}{2}} H_{n+\frac{1}{2}}^{(2)}(kr_2)}{r_1^{\frac{1}{2}} H_{n+\frac{1}{2}}^{(2)}(kr_1)} (1 + \epsilon(n, kr_1, kr_2)) \quad (11a)$$

where

$$\epsilon(n, kr_1, kr_2) = \frac{n(n+1)}{2k^2} \left(\frac{1}{r_2^2} - \frac{1}{r_1^2} \right) + O\left(\frac{1}{k^4 r^4}\right), \quad \text{as } r \rightarrow \infty \quad (11b)$$

with $r = \min(r_1, r_2)$.

We make three observations regarding this result:

1. To make the reciprocity between the nearfield and the farfield, we can take $r_1 = r < \infty$ and $r_2 = \infty$.
2. The quantities in (11a) are complex. However, the error $\epsilon(n, kr_1, kr_2)$ term is purely real, meaning the error is only in the magnitude or equivalently there is no error in the phase angle. This follows from the property $\arg(z_1/z_2) = \arg(z_2^*/z_1^*)$ where z_1 and z_2 are complex numbers.
3. Consider the tradeoff between operating at a distance (measured in wavelengths) sufficiently large to ensure the dominant error term in (11b) to be small. (For analysis purposes we take $r_1 = r$ and $r_2 = \infty$.) This requires, after taking the square root,

$$\sqrt{\frac{n(n+1)}{8\pi^2}} \ll \frac{r}{\lambda}. \quad (12)$$

3.2. Reciprocity Relationship

We now show how beampattern specification (analysis) at r_1 and resynthesis at r_2 relates to a conjugate beampattern specification (analysis) at r_2 and resynthesis at r_1 .

With a beampattern $b(\theta, \phi)$ specification given at radius r_1 the synthesised beampattern at distance r_2 is denoted and given by

$$\begin{aligned} b_{r_2}(\theta, \phi) \Big|_{b_{r_1}=b} &= \sum_{n=0}^{\infty} \zeta_n \mathcal{A}_n(b) \frac{r_2^{\frac{1}{2}} H_{n+\frac{1}{2}}^{(1)}(kr_2)}{r_1^{\frac{1}{2}} H_{n+\frac{1}{2}}^{(1)}(kr_1)} P_n(\cos \theta) \\ &+ \sum_{n=1}^{\infty} \frac{r_2^{\frac{1}{2}} H_{n+\frac{1}{2}}^{(1)}(kr_2)}{r_1^{\frac{1}{2}} H_{n+\frac{1}{2}}^{(1)}(kr_1)} \sum_{m=1}^n \zeta_n^m P_n^m(\cos \theta) \\ &\times (\mathcal{B}_{mn}(b) e^{jm\phi} + \mathcal{C}_{mn}(b) e^{-jm\phi}). \quad (13) \end{aligned}$$

This equation follows from substituting (8a)–(8c) in (7).

Compare this with a complex conjugate beampattern $b^*(\theta, \phi)$ specification at radius r_2 that has been synthesised

at r_1

$$\begin{aligned} b_{r_1}(\theta, \phi) \Big|_{b_{r_2}=b^*} &= \sum_{n=0}^{\infty} \zeta_n \mathcal{A}_n(b^*) \frac{r_1^{\frac{1}{2}} H_{n+\frac{1}{2}}^{(1)}(kr_1)}{r_2^{\frac{1}{2}} H_{n+\frac{1}{2}}^{(1)}(kr_2)} P_n(\cos \theta) \\ &+ \sum_{n=1}^{\infty} \frac{r_1^{\frac{1}{2}} H_{n+\frac{1}{2}}^{(1)}(kr_1)}{r_2^{\frac{1}{2}} H_{n+\frac{1}{2}}^{(1)}(kr_2)} \sum_{m=1}^n \zeta_n^m P_n^m(\cos \theta) \\ &\times (\mathcal{B}_{mn}(b^*) e^{jm\phi} + \mathcal{C}_{mn}(b^*) e^{-jm\phi}). \quad (14) \end{aligned}$$

Noting $\mathcal{A}_n(b^*) = \mathcal{A}_n^*(b)$, $\mathcal{B}_{mn}(b^*) = \mathcal{C}_{mn}^*(b)$ and $\mathcal{C}_{mn}(b^*) = \mathcal{B}_{mn}^*(b)$, and taking the complex conjugate of (14) yields

$$\begin{aligned} b_{r_1}^*(\theta, \phi) \Big|_{b_{r_2}=b^*} &= \sum_{n=0}^{\infty} \zeta_n \mathcal{A}_n(b) \frac{r_1^{\frac{1}{2}} H_{n+\frac{1}{2}}^{(2)}(kr_1)}{r_2^{\frac{1}{2}} H_{n+\frac{1}{2}}^{(2)}(kr_2)} P_n(\cos \theta) \\ &+ \sum_{n=1}^{\infty} \frac{r_1^{\frac{1}{2}} H_{n+\frac{1}{2}}^{(2)}(kr_1)}{r_2^{\frac{1}{2}} H_{n+\frac{1}{2}}^{(2)}(kr_2)} \sum_{m=1}^n \zeta_n^m P_n^m(\cos \theta) \\ &\times (\mathcal{C}_{mn}(b) e^{-jm\phi} + \mathcal{B}_{mn}(b) e^{jm\phi}) \quad (15) \end{aligned}$$

where $r = \min(r_1, r_2)$ (alternatively for $r_1 \rightarrow r_2$ this also holds). Thus, using Proposition 1 we have established:

Proposition 2 *Let λ be the wavelength and $k = 2\pi/\lambda$ the wave number, then*

$$b_{r_1}^*(\theta, \phi) \Big|_{b_{r_2}=b^*} = b_{r_2}(\theta, \phi) \Big|_{b_{r_1}=b} \left(1 + O\left(\frac{1}{k^2 r_2^2} - \frac{1}{k^2 r_1^2}\right) \right), \quad \text{as } r \rightarrow \infty \quad (16)$$

with $r = \min(r_1, r_2)$.

We make the following observations:

1. If $r_2 = \infty$ then this result is saying that a nearfield problem can be solved approximately by solving a related farfield problem.
2. The reciprocity holds whenever the dominant error term can be made small which implies either the beampattern is low-pass in character, i.e., most of the energy is in the lower order modes (small n , which generally holds), or the difference in the radial distances, $r_1 - r_2$ is small enough.
3. In the cases where accuracy in the reciprocity is in question a farfield technique which acts to replace (16) with an exact relation can be developed (see [3]) at the cost of considerable computation effort.

4. NEARFIELD DESIGN PROCEDURE

The reciprocity relationship (16) with $r_1 = r$ and $r_2 = \infty$, leads to the corollary of Proposition 1:

Proposition 3 *The farfield beampattern corresponding to a desired nearfield beampattern specification $b_r(\theta, \phi) = b(\theta, \phi)$ satisfies the asymptotic equivalence*

$$b_{\infty}(\theta, \phi) \Big|_{b_r=b} \sim b_r^*(\theta, \phi) \Big|_{b_{\infty}=b^*}, \quad \text{as } r \rightarrow \infty. \quad (17)$$

By assuming (17) holds with equality we have the following design procedure:

Nearfield Design Procedure

1. Design for $b^*(\theta, \phi)$ in the farfield, i.e., $b_\infty(\theta, \phi) = b^*(\theta, \phi)$
2. Using the design in Step 1 determine the beam-pattern response at distance r . Call this $a(\theta, \phi)$, i.e., $a(\theta, \phi) = b_r(\theta, \phi)|_{b_\infty=b^*}$.
3. Design for $a^*(\theta, \phi)$ in the farfield.

This procedure requires a nearfield beampattern determination from farfield data, sandwiched between two farfield designs.

5. DESIGN EXAMPLE

The following example shows the result of this design procedure in comparison with a technique developed in [2]. The objective was to realize a seventh-order zero-phase Chebyshev 25 dB beampattern in the nearfield at a radius of 3 wavelengths. The array sensors are colinear and aligned along the azimuth axis of rotation.

Step 1 of the design procedure in Section 4 required a design to realize the complex conjugate of this Chebyshev beampattern in the farfield. This is a classical design problem [5], and the weights for a 7 sensor half-wavelength spaced farfield array are easily calculated. The resulting farfield beampattern is $b^*(\theta, \phi)$ in the design procedure, i.e., the complex conjugate of the objective beampattern.

The response of this farfield beamformer was then evaluated in the nearfield at the required radius of 3 wavelengths. Figure 1(a) shows the resulting beampattern. This is $a(\theta, \phi)$ in Step 2 of Section 4.

Step 3 of the design procedure required designing a farfield beamformer to realize $a^*(\theta, \phi)$. We used a weighted complex-valued least-squares design method [6] to realize $a^*(\theta, \phi)$ with a 13 element quarter-wavelength spaced array. Angles outside the range 70° – 110° were weighted more heavily so that the sidelobe region of the desired Chebyshev beampattern would be accurately approximated. The resulting farfield realization is shown in Fig. 1(b).

Finally, to verify the design objectives had been met, this beamformer was simulated in the nearfield at a radius of 3 wavelengths; the nearfield beampattern shown solid in Fig. 1(c) resulted. Also shown is the desired Chebyshev 25 dB beampattern (dotted), and the response of the nearfield method of [2] (dashed). We note that the proposed nearfield design technique provides a very close realization of the desired beampattern over all angles, not just at angles close to broadside as for the nearfield method of [2].

This example highlights the main feature of our proposed nearfield beamforming procedure: when the reciprocity relation holds, it is only necessary to use well-established farfield beamformer design techniques in the design of a nearfield beamformer.

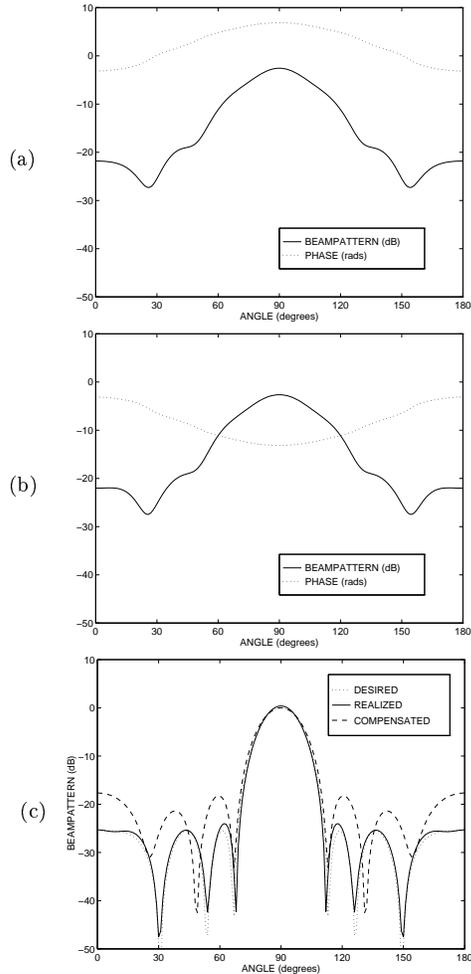


Figure 1: Demonstration of Nearfield/Farfield Reciprocity. (a) Result of using a Chebyshev 25 dB farfield beamformer in the nearfield at a radius of 3 wavelengths. (b) Farfield realization of conjugate beampattern using least square design for a 13 quarter-wavelength spaced array. (c) Result of using 13 sensor farfield beamformer in the nearfield.

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