# A JOINT DETECTION-ESTIMATION SCHEME FOR THE ANALYSIS OF NOISY COMPLEX SINUSOIDS

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## ABSTRACT

**Classical high resolution (HR) spectral analysis methods for the estimation of complex sinusoids parameters require the a priori knowledge of the number of sinusoids. In most situations, the correctness of this knowledge is crucial. In this paper, we present a new HR subspace method. Its novelty stems from the fact that the analysis of complex sinusoids is considered as a joint "detection-estimation" issue. Simulation results and an application on real radar signals are presented to illustrate the efficiency of this method.**

## 1. INTRODUCTION

Numerous High Resolution (HR) subspace methods [1] have been proposed to estimate either the frequencies of sinusoids embedded in noise, or the directions of arrivals of sources. They are developed under the hypothesis that the signal subspace rank (ie the number of sinusoids or sources) is known. They present a lack of robustness with respect to the misreading of this information. In real applications however, the signal subspace rank is generally unavailable. So, one needs, in a detection step prior to the estimation one, to determine the number of sources. The estimation accuracy depends therefore on the detection efficiency. The detection step which is based on the use of criteria like Minimum Description Length (MDL) [2], is known to be less efficient than subspace methods, particularly for low SNR. In order to overcome this problem, P. Duvaut [3] proposes to reformulate the problem of the analysis of complex sinusoids into a joint "detection-estimation" issue and derived the Expulse algorithm based on the deconvolution of the periodogram. The main drawback of this algorithm is that it requires a careful tuning on several parameters. In this paper, we propose, in the framework of the joint "detection-estimation" approach, a new subspace based method. This method does not require tuning of any parameters.

## 2. NOTATION AND PROBLEM FORMULATION

Let us consider the class of discrete-time processes $y(m)$ which can be modeled as a sum of $p$ complex exponentials corrupted with a zero-mean white gaussian circular noise $n(m)$ with variance $\sigma_n^2$ :

$$\begin{aligned} y(m) &= x(m) + n(m) \\ &= \sum_{i=1}^{p} a_i \exp(j2\pi f_i^e m + j\phi_i) + n(m) \end{aligned}$$

where $\{f_i^e\}_{1\le i\le p}$ is the set of the frequencies to be estimated, $\{a_i\}_{1\le i\le p}$ the set of constant amplitudes and $\{\phi_i\}_{1\le i\le p}$ the set of initial phases. Consider the ensemble-averaged correlation matrix $\mathbf{R}_y$ of $y(m)$ :

$$\mathbf{R}_y = \mathcal{E}\left\{\mathbf{y}(m)\mathbf{y}^H(m)\right\} = \mathbf{S}_{\mathbf{f}^e}\mathbf{A}\mathbf{S}_{\mathbf{f}^e}^H + \sigma_n^2\mathbf{I}_L$$

$$\text{where}\left\{\begin{array}{c} \mathbf{y}(m) = \left[\begin{array}{ccc} y(m) & \ldots & y(m-L+1) \end{array}\right]^T \\ \mathbf{A} = \mathbf{diag}\left(|a_i|^2\right) \end{array}\right.$$

We define the Vandermonde matrix as :

$$\mathbf{S}_{\mathbf{f}^e} = \left[\begin{array}{ccc} \mathbf{s}(f_1^e) & \ldots & \mathbf{f}(f_p^e) \end{array}\right]$$

with

$$\mathbf{s}(f_i^e) = \left[\begin{array}{cccc} 1 & \exp(j2\pi f_i^e) & \ldots & \exp(j2\pi f_i^e(L-1)) \end{array}\right]^T$$

Let us define the signal subspace by :

$$U_S = Span\left\{(\mathbf{s}(f_i^e))_{1\le i\le p}\right\}$$

Denoting the eigenvalues of $\mathbf{R}_y$ in increasing order by $(\lambda_i)_{1\le i\le p}$ and their associated eigenvectors by $(\mathbf{u}_i)_{1\le i\le p}$, we have :

$$\begin{aligned} &\hookrightarrow \left\{\begin{array}{lll} \lambda_i & > \sigma_n^2 & 1 \le i \le p \\ & = \sigma_n^2 & p+1 \le i \le L \end{array}\right. \\ &\hookrightarrow Span\left\{(\mathbf{u}_i)_{1\le i\le p}\right\} = U_S \end{aligned}$$

We define the noise subspace $U_N$ as the subspace spanned by $L-p$ eigenvectors associated to $\sigma_n^2$. Since $\mathbf{R}_y$ is an hermitian matrix, $U_S$ and $U_N$ are orthogonal subspaces.

## 3. A NEW CRITERION

The main idea of the method that we introduce for estimating the frequencies is to identify the best Maximum Likelihood model $w(n)$ which verifies :

$$w(m) = \sum_{i=1}^{p(\mathbf{f})} \sqrt{\sigma^2} \exp(j2\pi f_i m) + \tilde{n}(m)$$

where $\tilde{n}(m)$ is a zero-mean white gaussian circular noise with variance $\tilde{\sigma}_n^2$ and $p(\mathbf{f})$ is the number of components of $\mathbf{f} = \begin{bmatrix} f_1 & \dots & f_{p(\mathbf{f})} \end{bmatrix}^{\mathbf{T}}$.
The powers $\tilde{\sigma}_n^2$ and $\sigma^2$ are defined so that the signal $y(m)$ and the model $w(m)$ have the same powers and the same ensemble-averaged correlation matrix determinants. Consequently, the criterion we introduce is the following one. Let us note $g$ the probability density function of the observation vectors $\{\mathbf{y(i)}\}_{\mathbf{1\leq i\leq N-L+1}}$, the estimated frequency vector is the one which maximizes the cost function defined by :

$$\begin{aligned} C(\mathbf{f}) &= \ln\left(g(\mathbf{y}_1, \cdots, \mathbf{y}_{N-L+1}; \mathbf{f})\right) \\ &= -L(N-L+1)\ln(\pi) \\ &\quad -(N-L+1)\ln\left(|\mathbf{R}_w(\mathbf{f})|\right) - tr(\mathbf{R}_w^{-1}(\mathbf{f})\mathbf{H}_y\mathbf{H}_y^H) \end{aligned}$$

$$\text{where} \left\{ \begin{array}{c} \mathbf{H}_y = [\mathbf{y}_1, \cdots, \mathbf{y}_{N-L+1}] \\ \mathbf{R}_w(\mathbf{f}) = \sigma^2(\mathbf{f})\, \mathbf{S_f S_f}^H + \tilde{\sigma}_n^2(\mathbf{f})\, \mathbf{I}_L \end{array} \right.$$

$$\text{with} \left\{ \begin{array}{c} \mathbf{y}(m) = \begin{bmatrix} y(m) & \dots & y(m+L-1) \end{bmatrix}^T \\ \mathbf{S_f} = \left[\mathbf{s}(f_1), \cdots, \mathbf{s}(f_{p(\mathbf{f})})\right] \end{array} \right.$$

By removing the terms which are independent of $\mathbf{f}$, the criterion can be simplified in :

$$C(\mathbf{f}) = -tr(\mathbf{R}_w^{-1}(\mathbf{f})\mathbf{H}_y\mathbf{H}_y^H)$$

## 4. ESTIMATION OF $\sigma^2$ AND $\tilde{\sigma}_N^2$

As mentioned in section 2, $\tilde{\sigma}_n^2(\mathbf{f})$ and $\sigma^2(\mathbf{f})$ are defined so that the signal $y(m)$ and the model $w(m)$ have the same powers and the same ensemble-averaged correlation matrix determinants. Using the equality between the powers of $y(m)$ and $w(m)$, we get :

$$P_y = P_w = p(\mathbf{f})\sigma^2(\mathbf{f}) + \tilde{\sigma}_n^2(\mathbf{f})$$

Define $\alpha(\mathbf{f}) = \frac{\sigma^2(\mathbf{f})}{\tilde{\sigma}_n^2(\mathbf{f})}$, we thus can write :

$$P_y = \tilde{\sigma_n}^2(\mathbf{f})\,(p(\mathbf{f})\alpha(\mathbf{f}) + 1) \tag{1}$$

The correlation matrix of $w(m)$ can be written as follows :

$$\mathbf{R}_w(\mathbf{f}) = \tilde{\sigma}_n^2(\mathbf{f})\ \left(\alpha(\mathbf{f})\mathbf{S_f S_f}^H + \mathbf{I}_L\right)$$

Now the use of the equality between the determinants of $\mathbf{R}_y$ and $\mathbf{R}_w$ yields :

$$\begin{aligned} |\mathbf{R}_y| = |\mathbf{R}_w| &= \tilde{\sigma}_n^{2^L}(\mathbf{f}) \left|\alpha(\mathbf{f})\mathbf{S_f S_f}^H + \mathbf{I}_L\right| \\ &= \tilde{\sigma}_n^{2^L}(\mathbf{f}) \left|\alpha(\mathbf{f})\mathbf{S}^H(\mathbf{f})\mathbf{S}(\mathbf{f}) + \mathbf{I}_{p(\mathbf{f})}\right| \end{aligned}$$

Let $\gamma_1(\mathbf{f}), \cdots, \gamma_{p(\mathbf{f})}(\mathbf{f})$ be the eigenvalues of the matrix $\mathbf{S}^H(\mathbf{f})\mathbf{S}(\mathbf{f})$. One find :

$$|\mathbf{R}_y| = \tilde{\sigma}_n^{2^L}(\mathbf{f}) \prod_{i=1}^{p(\mathbf{f})} (1 + \gamma_i(\mathbf{f})\alpha(\mathbf{f}))$$

By substituting $\tilde{\sigma}_n^2(\mathbf{f})$ by its value deduced from Eq (1), it is straightforward to demonstrate that $\alpha(\mathbf{f})$ is a root of :

$$P(\alpha) = \prod_{i=1}^{p(\mathbf{f})} (1 + \gamma_i(\mathbf{f})\alpha)\, P_y^L - |\mathbf{R}_y|\,(p(\mathbf{f})\alpha(\mathbf{f}) + 1)^L$$

**Lemma : The polynomial $P$ has one and only one real positive root.**

If we define $\alpha_r(\mathbf{f})$ as the unique real positive root of $P$, and use Eq (2), we obtain the estimates of $\sigma^2(\mathbf{f})$ and $\tilde{\sigma}_n^2(\mathbf{f})$ :

$$\tilde{\sigma}_n^2(\mathbf{f}) = \frac{P_y}{p(\mathbf{f})\alpha_r(\mathbf{f}) + 1} \text{ and } \sigma^2(\mathbf{f}) = \frac{P_y \alpha_r(\mathbf{f})}{p(\mathbf{f})\alpha_r(\mathbf{f}) + 1}$$

## 5. FREQUENCY ESTIMATION

### 5.1. Reduction of the sphere of investigation

It is impossible to exactly maximize the $C$ function and then find the appropriate $\mathbf{f}$. So we propose to track the solution by estimating a set of $K$ frequencies which contains the true frequencies $\{f_i^e\}_{1\leq i\leq p}$. This naturally implies that $K \geq p$. We will note $\mathfrak{F} = \{\mathfrak{f}_\mathfrak{i}\}_{1\leq \mathfrak{i}\leq \mathfrak{K}}$ this set of $K$ frequencies and will call it the set of the *possible frequencies*. $\mathfrak{F}$ can be obtained by applying an algorithm like MUSIC or ESPRIT with the order $K$. Indeed it can be shown, in the asymptotical case, that the set $\mathfrak{F}$ obtained by one of these two algorithms contains the true frequencies of the signal $y(m)$.

### 5.2. An iterative scheme

The maximization of the $C$ criterion on the $2^K$ permutations on the set $\mathfrak{F}$ can be burdensome. In this paragraph, we thus propose the following iterative scheme :

*initialization*
$p = 0$
$C_{max} = -\infty$
$f_{d_0} = \{\}$
*iterations*
*for* $i = 1$ *to* $K$
$\quad f_{d_i} = \underset{f_j \in \mathfrak{F} \backslash \{f_{d_1}, \cdots, f_{d_{i-1}}\}}{\arg\max} C([f_{d_1}, \cdots, f_{d_{i-1}}, f_j]^T)$
$\quad$ *if* $C([f_{d_1}, \cdots, f_{d_i}]^T) > C_{max}$
$\quad\quad C_{max} \leftarrow C([f_{d_1}, \cdots, f_{d_i}]^T)$
$\quad\quad p \leftarrow p + 1$
$\quad$ *end*
*end*
$\hat{f} = ([f_{d_1}, \cdots, f_{d_p}]^T))$

## 6. OPTIMIZATION

The variances on the estimates of $\{f_i^e\}_{1 \leq i \leq p}$ contained in the set $\mathfrak{F}$ is greater than the ones obtained by using MUSIC or ESPRIT with the exact order. So, we propose a final stage in deriving to this method which is a gradient based optimization :

$$\mathbf{f}_{n+1} = \mathbf{f}_n - \xi_n \left(\frac{\partial C}{\partial \mathbf{f}}\right)_{\mathbf{f}=\mathbf{f}_n}$$

where $\xi_n$ is a small real positive constant and $\mathbf{f}_0$ is the frequency vector $\widehat{\mathbf{f}}$ estimated by the method introduced in section 5.

## 7. SIMULATIONS

In this section, we present simulation results to demonstrate the efficiency of the proposed method. The data model was generated from :

$$y(m) = \sqrt{10}\exp(j2\pi f_1 m) + \exp(j2\pi f_2 m + \phi) + n(m)$$

choosing the normalized frequencies $f_1$ and $f_2$ as 0.28 and $0.28 + \delta$ with $\delta = 1/N$. The number of samples $N$ was chosen as 25, the correlation matrix size $L$ as 12 and the overestimation order $K$ as $L-1$. The variance of the noise $\sigma_n^2$ is set so as to give the desired SNR defined as $SNR = 10\log(\sum_{i=1}^{p} |a_i|^2)/(\sigma_n^2)$. The simulation results were obtained from 100 Monte Carlo trials. In each trial, the noise realization and the initial phase $\phi$ are chosen randomly. Figure 1 shows the mean square error (MSE) of $f_1$ and $f_2$ estimated respectively with the proposed and with MUSIC methods. Figure 2 shows the probability of error respectively for the proposed and for MDL methods.

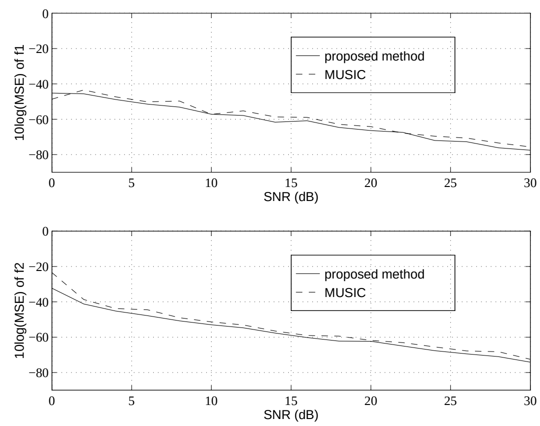


Figure 1: MSE's of the estimates of $f_1$ and $f_2$ versus SNR with MUSIC and with the proposed method, N=25, L=12

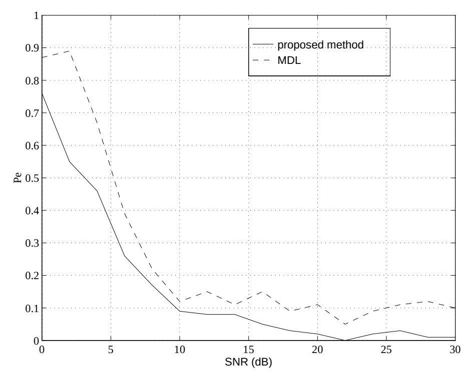


Figure 2: Probability of detection errors versus SNR for MDL and the proposed method, N=25, L=12

## 8. APPLICATION TO REAL DATA

We now illustrate the efficiency of the algorithm we have introduced with real data. This illustration deals with the problem of localization of the scattering centers of a target. When illuminated by an incident plane wave at frequency $f$, we can model the amplitude of a radar target echo as follows [4] :

$$C(f) = \sum_{i=1}^{p} a_i e^{-j2\pi\frac{2f}{c}x_i}$$

where $p$ is the number of scatterers of the target, $a_i$ the amplitude of the $i^{th}$ scatterer and $x_i$ the projected distance of this scatterer on the radar line of sight.

By measuring the amplitude of the radar target echo for a set of frequencies $\{f_i\}_{1 \leq i \leq N}$ (not to confuse with the frequencies of the model introduced in section 2), we obtain some estimates of the projected distances of the scatterers on the radar line of sight by using the FFT or high resolution spectral analysis algorithms. The following example shows the results given by the FFT, MUSIC and the algorithm we have introduced on real measurements. The frequency step is $0.0384\ GHz$.

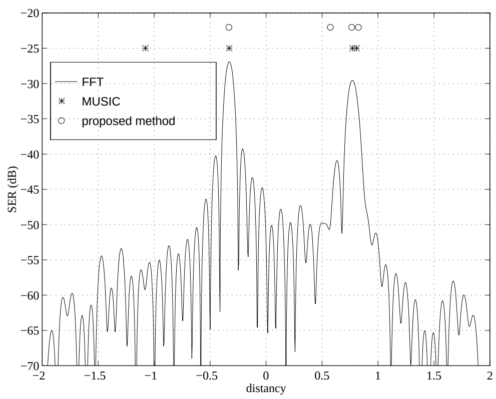


Figure 3: Simplified target : frequency band 8.2-10 GHz

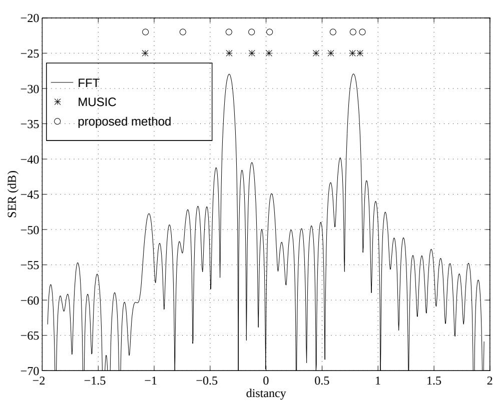


Figure 4: Simplified target : frequency band 16.2-18 GHz

## 9. CONCLUSION

A new spectral analysis method based on subspace decomposition has been proposed. This method is based on a joint "detection-estimation" scheme. This method is carried out in three steps. First, we obtain a set of possible frequencies which contains some estimates of the true signal frequencies. The true estimates are then detected with a ML based method. Finally, a gradient-like optimization provides more accurate estimates of the signal frequencies. We have shown through simulations that the proposed method gives better results than the ones based on the classical "detection then estimation" scheme. We have illustrated the efficiency of this method on real radar measurements.

## 10. REFERENCES


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