RECURSIVE EIGENDECOMPOSITION VIA AUTOREGRESSIVE ANALYSIS & AGO-ANTAGONISTIC REGULARIZATION

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1. ABSTRACT

A new recursive eigendecomposition algorithm of Complex Hermitian Tœplitz matrices is studied. Based on Trench's inversion of Tœplitz matrices from their autoregressive analysis, we have developed a fast recursive iterative algorithm that takes into account the rank-one modification of successive order Toeplitz matrices. To speed up the computational time and to increase numerical stability of ill-conditioned eigendecomposition in case of very short data records analysis, we have extended this method by introducing an ago-antagonistic regularized reflection coefficient via Levinson equation. We provide a geometrical interpretation of this new recursive eigendecomposition.

2. PREAMBLE

Let us remind you that Levinson algorithm provides Cholesky factorization of the inverse Tœplitz matrix. Rank-one modification approach leads to the Gohberg-Semencul formula which is an integrated version of Trench algorithm [5]. Trench algorithm induces an order recursive structure of the inverse Tœplitz matrix. We propose to exploit this existing structure to achieve a fast and robust eigendecomposition. First, we obtain eigenvalues by finding the roots of an autoregressive parameters-based function [2]. At each order, a number of independent structurally identical nonlinear problems is solved in parallel. Derivative of this intermediate function is geometrically interpreted. In a second step, via Levinson equation, reflection coefficient is used to decrease computational complexity and increase stability by an ago-antagonistic regularization [1][2]. Ago-antagonism [6], conceived as Minimum Free Enthalpy concept in a thermodynamic analogy approach, extends regularization method and avoids over-regularization problems. Among research in the area of recursive eigenspace decomposition, other algorithms have been proposed taking advantage of direct Tœplitz matrix structure, like RISE [3][4], but they are not very well adapted to very short data records analysis.

3. RECURSIVE EIGENDECOMPOSITION VIA AUTOREGRESSIVE ANALYSIS

3.1 Yule-Walker and Levinson Equation

Autoregressive analysis problem is solved by Yule-Walker equation. Order recursive structure of Tœplitz correlation matrix provides the recursive Levinson equation :

$$R_n.A_n = -C_n \quad \text{with} \quad R_n = \begin{bmatrix} c_0 & C_{n-1}^{+} \\ C_{n-1} & R_{n-1} \end{bmatrix} = \begin{bmatrix} R_{n-1} & C_{n-1}^{(-)} \\ C_{n-1}^{(-)+} & c_0 \end{bmatrix}$$

where

$$C_n = \begin{bmatrix} c_1 \\ c_2 \\ \ldots \\ c_n \end{bmatrix} \quad , \quad c_k = E\left[x_n.x_{n-k}^{*}\right] \quad \text{and} \quad A_n = \begin{bmatrix} a_1^{(n)} \\ a_2^{(n)} \\ \ldots \\ a_n^{(n)} \end{bmatrix}$$

with the following notation : $V^{(-)} = J.V^{*}$

where J is an anti-diagonal matrix. Then, Levinson Equation is given by :

$$A_n = \begin{bmatrix} A_{n-1} \\ 0 \end{bmatrix} + \mu_n.\begin{bmatrix} A_{n-1}^{(-)} \\ 1 \end{bmatrix} \quad \text{where} \quad \mu_n = a_n^{(n)} \tag{1}$$

3.2 Cholesky, Trench and Gohberg-Semencul Equation

Trench has found order recursive structure of the inverse correlation Tœplitz matrix via autoregressive parameters :

$$R_n^{-1} = \Phi_n = \begin{bmatrix} \alpha_{n-1} & \alpha_{n-1}.A_{n-1}^{+} \\ \alpha_{n-1}.A_{n-1} & \Phi_{n-1} + \alpha_{n-1}.A_{n-1}.A_{n-1}^{+} \end{bmatrix} \tag{2}$$

or $$R_n^{-1} = \Phi_n = \begin{bmatrix} 0 & 0_{n-1xn-1}^{+} \\ 0_{n-1xn-1} & \Phi_{n-1} \end{bmatrix} + \alpha_{n-1}.T_{n-1}.T_{n-1}^{+}$$

where : $\alpha_n^{-1} = \left[1-|\mu_n|^2\right].\alpha_{n-1}^{-1}$ and $T_{n-1} = \begin{bmatrix} 1 \\ A_{n-1} \end{bmatrix}$

It prooves that Levinson algorithm correponds to the Cholesky factorization of $\Phi_n = R_n^{-1}$:

$$R_n^{-1} = \sum_{k=0}^{n-1} \alpha_k.T_k.T_k^{+} = B_n.\Gamma_n.B_n^{+}$$

where :

$$B_n = \begin{bmatrix} Y_n^{(1)} & \ldots & Y_n^{(n)} \end{bmatrix} \ , \ Y_n^{(k)} = \begin{bmatrix} 0_{k-1} \\ 1 \\ A_{n-k} \end{bmatrix} \quad \text{and} \quad \Gamma_n = \mathrm{diag}\{\alpha_{n-1},\ldots,\alpha_0\}$$

Adding a rank-one modification to an Hermitian matrix has the same effect as appending a column to the triangular matrix of its Cholesky factorization. In the same way, Trench has identified an other equivalent matrix structure of the inverse Tœplitz correlation matrix :

$$R_{n-1}^{-1} = \Phi_n = \begin{bmatrix} \Phi_{n-1} + \alpha_{n-1}.A_{n-1}^{(-)}.A_{n-1}^{(-)+} & \alpha_{n-1}.A_{n-1}^{(-)} \\ \alpha_{n-1}.A_{n-1}^{(-)+} & \alpha_{n-1} \end{bmatrix} \tag{3}$$

If we consider rank-one modification from one order to the next, we find the Gohberg-Semencul formula :

Let : $$Z_n = \begin{bmatrix} 0 & \ldots & \ldots & 0 & 0 \\ 1 & 0 & \ldots & \ldots & 0 \\ 0 & 1 & \ldots & \ldots & \ldots \\ \ldots & \ldots & 1 & 0 & \\ 0 & \ldots & 0 & 1 & 0 \end{bmatrix}$$

$$\nabla R_n^{-1} = R_n^{-1} - Z_n.R_n^{-1}.Z_n^+ = \alpha_{n-1}.\left[T_{n-1}.T_{n-1}^+ - \left(Z_n.T_{n-1}^{(-)}\right).\left(Z_n.T_{n-1}^{(-)}\right)^+\right]$$

Let : $W_n = \sqrt{\alpha_n}.T_n$

$$\nabla R_n^{-1} = R_n^{-1} - Z_n.R_n^{-1}.Z_n^+ = W_{n-1}.W_{n-1}^+ - \left(Z_n.W_{n-1}^{(-)}\right).\left(Z_n.W_{n-1}^{(-)}\right)^+$$

After n steps :

$$\nabla^k R_n^{-1} = \left(Z_n^k.W_{n-1}\right).\left(Z_n^k.W_{n-1}\right)^+ - \left(Z_n^{k+1}.W_{n-1}^{(-)}\right).\left(Z_n^{k+1}.W_{n-1}^{(-)}\right)^+$$

It leads to the following equation, that is an integrated Trench Algorithm version, known as Gohberg-Semencul formula.

$R_n^{-1} = Q_n.Q_n^+ - K_n.K_n^+$ with $Q_n = \left[W_{n-1} \quad Z_n.W_{n-1} \quad \dots \quad Z_n^{n-1}.W_{n-1}\right]$

and $K_n = \left[Z_n.W_{n-1}^{(-)} \quad Z_n^2.W_{n-1}^{(-)} \quad \dots \quad Z_n^n.W_{n-1}^{(-)}\right]$

3.3 Recursive Eigendecomposition

Our algorithm uses rank-one modification structure of the successive inverse Tœplitz matrix to provide a recursive eigendecomposition :

$$\begin{cases}\Phi_n = R_n^{-1} = \begin{bmatrix}\alpha_{n-1} & \alpha_{n-1}.A_{n-1}^+ \\ \alpha_{n-1}.A_{n-1} & \Phi_{n-1} + \alpha_{n-1}.A_{n-1}.A_{n-1}^+\end{bmatrix} \\ \Phi_n.X_k^{(n)} = \eta_k^{(n)}.X_k^{(n)} \quad \text{with} \quad X_k^{(n)} = \begin{bmatrix}X_{k,1}^{(n)} \\ \underline{X}_k^{(n)}\end{bmatrix}\end{cases}$$

$$\Rightarrow \begin{cases}\alpha_{n-1}.T_{n-1}^+.X_k^{(n)} = \eta_k^{(n)}.X_{k,1}^{(n)} \\ A_{n-1}.\left[\alpha_{n-1}.T_{n-1}^+.X_k^{(n)}\right] + \left(\Phi_{n-1} - \eta_k^{(n)}.I_{n-1}\right).\underline{X}_k^{(n)} = 0\end{cases} \qquad (4)$$

If we assume that eigenvectors and eigenvalues at previous order are known :

$$\begin{cases}U_{n-1} = \left[X_1^{(n-1)} \quad \dots \quad X_{n-1}^{(n-1)}\right] \quad \text{with} \quad U_{n-1}^+.U_{n-1} = U_{n-1}.U_{n-1}^+ = I_{n-1} \\ U_{n-1}^+.\Phi_{n-1}.U_{n-1} = \Lambda_{n-1} = \text{diag}\left\{\dots, \eta_k^{(n-1)}, \dots\right\}\end{cases}$$

Then, eigenvalues are recursively provided by roots of function $F^{(n)}$, and eigenvectors can be computed by (6) :

$$\begin{cases}F^{(n)}\left(\eta_k^{(n)}\right) = \eta_k^{(n)} - \alpha_{n-1} + \alpha_{n-1}.\eta_k^{(n)}.\sum_{i=1}^{n-1}\dfrac{\left|A_{n-1}^+.X_i^{(n-1)}\right|^2}{\left(\eta_i^{(n-1)} - \eta_k^{(n)}\right)} = 0 \qquad (5) \\ X_k^{(n)} = \begin{bmatrix}X_{k,1}^{(n)} \\ -\eta_k^{(n)}.X_{k,1}^{(n)}.U_{n-1}.\left(\Lambda_{n-1} - \eta_k^{(n)}.I_{n-1}\right).U_{n-1}^+.A_{n-1}\end{bmatrix} \qquad (6)\end{cases}$$

If we apply corollaire of Courant-Fisher theorem, it proves the interlacing of eigenvalues at successive orders, because inverse correlation matrix Φ_{n-1} is included in Φ_n. We also know that the inverse eigenvalues are all positive and inferior to the inverse prediction error power α_n :

$$0 < \eta_n^{(n)} < \eta_{n-1}^{(n-1)} < \eta_{n-1}^{(n)} < \dots < \eta_2^{(n)} < \eta_1^{(n-1)} < \eta_1^{(n)} < \alpha_n$$

The interlacing structure of the inverse eigenvalues simplifies research of $F^{(n)}$ roots because derivative of this function is strictly greater than unity :

$$\frac{\partial F^{(n)}(\eta)}{\partial \eta} = 1 + \alpha_{n-1}.\sum_{k=1}^{n-1}\frac{\eta_k^{(n-1)}.\left|A_{n-1}^+.X_k^{(n-1)}\right|^2}{\left(\eta_k^{(n-1)} - \eta\right)^2} > 1 \qquad (7)$$

Our algorithm is reduced to n parallel researches of one root of $F^{(n)}(.)$ on each interval $\left]\eta_{k+1}^{(n-1)}, \eta_k^{(n-1)}\right[$.

Recursive structure of the inverse Tœplitz matrix allows to obtain a new equation about derivative of $F^{(n)}$:

$\eta_k^{(n)} = X_k^{(n)+}.\Phi_n.X_k^{(n)}$ with $X_i^{(n)+}.X_k^{(n)} = \delta_{i,k}$ but if we use (2) :

$$\eta_i^{(n)} = X_i^{(n)+}.\left(\begin{bmatrix}0 & 0_{n-1}^+ \\ 0_{n-1} & \Phi_{n-1}\end{bmatrix} + \alpha_{n-1}.T_{n-1}.T_{n-1}^+\right).X_i^{(n)}$$

$$\Rightarrow \frac{\partial F^{(n)}\left(\eta_k^{(n)}\right)}{\partial \eta} = \frac{\alpha_{n-1}}{\eta_k^{(n)}.\left|X_{k,1}^{(n)}\right|^2} \qquad (8)$$

In the same way, expression (3) provides :

$$\alpha_{n-1}.T_{n-1}^{(-)+}.X_k^{(n)} = \eta_k^{(n)}.X_{k,n}^{(n)} \qquad (9)$$

and $$X_k^{(n)} = \begin{bmatrix}-\eta_k^{(n)}.X_{k,n}^{(n)}.\left(\Phi_{n-1} - \eta_k^{(n)}\right)^{-1}.A_{n-1}^{(-)} \\ X_{k,n}^{(n)}\end{bmatrix} \qquad (10)$$

eigenvalues are also provided by roots of :

$$G^{(n)}\left(\eta_k^{(n)}\right) = \eta_k^{(n)} - \alpha_{n-1} + \alpha_{n-1}.\eta_k^{(n)}.\sum_{i=1}^{n-1}\frac{\left|A_{n-1}^{(-)+}.X_i^{(n-1)}\right|^2}{\left(\eta_i^{(n-1)} - \eta_k^{(n)}\right)} = 0 \qquad (11)$$

with $$\frac{\partial G^{(n)}\left(\eta_k^{(n)}\right)}{\partial \eta} = \frac{\alpha_{n-1}}{\eta_k^{(n)}.\left|X_{k,n}^{(n)}\right|^2} \qquad (12)$$

4. GEOMETRICAL INTERPRETATION

4.1 Projection Interpretation

By identification of these two following expressions of the inverse correlation Tœplitz matrix Φ_n, we have :

$$\Phi_n = \begin{bmatrix}\alpha_{n-1} & \alpha_{n-1}.A_{n-1}^+ \\ \alpha_{n-1}.A_{n-1} & \Phi_{n-1} + \alpha_{n-1}.A_{n-1}.A_{n-1}^+\end{bmatrix} = \sum_{k=1}^{n}\eta_k^{(n)}.X_k^{(n)}.X_k^{(n)+}$$

$$\alpha_{n-1} = \sum_{k=1}^{n}\eta_k^{(n)}.\left|X_{k,1}^{(n)}\right|^2 \quad \text{and} \quad T_{n-1} = \sum_{k=1}^{n}\frac{\eta_k^{(n)}.X_{k,1}^{(n)*}}{\alpha_{n-1}}.X_k^{(n)}$$

From equation (8), we deduce a geometrical relation :

$$\sum_{k=1}^{n}\left(\frac{\partial F^{(n)}\left(\eta_k^{(n)}\right)}{\partial \eta}\right)^{-1} = 1 \quad \text{and} \quad T_{n-1} = \sum_{k=1}^{n}\left(\frac{\partial F^{(n)}\left(\eta_k^{(n)}\right)}{\partial \eta}\right)^{-1}.\frac{X_k^{(n)}}{X_{k,1}^{(n)}} \qquad (13)$$

In the Hilbert Space, the inverse derivative of $F^{(n)}(\eta_k^{(n)})$ appears as the projection of vector $[1\ A_{n-1}]^T$ (AR prediction vector) on eigenvector $X_k^{(n)}$, normalized by its first component $X_{i,1}^{(n)}$:

$$\left(\frac{\partial F^{(n)}\left(\eta_k^{(n)}\right)}{\partial \eta}\right)^{-1} = \left|X_{k,1}^{(n)}\right|^2.T_{n-1}^+.\frac{X_k^{(n)}}{X_{k,1}^{(n)}} = \left|X_{k,1}^{(n)}\right|^2.\left\langle T_{n-1}, \frac{X_k^{(n)}}{X_{k,1}}\right\rangle$$

with $\langle .,. \rangle$: inner product

In the same way, we have :

$$\sum_{k=1}^{n}\left(\frac{\partial G^{(n)}\left(\eta_k^{(n)}\right)}{\partial \eta}\right)^{-1} = 1 \quad \text{and} \quad T_{n-1}^{(-)} = \sum_{k=1}^{n}\left(\frac{\partial G^{(n)}\left(\eta_k^{(n)}\right)}{\partial \eta}\right)^{-1}.\frac{X_k^{(n)}}{X_{k,n}^{(n)}} \qquad (14)$$

4.2 Additional results

By using (13) and (6), we proove a new geometrical result :

$$\Rightarrow \sum_{k=1}^{n}\left(\frac{\partial F^{(n)}\left(\eta_k^{(n)}\right)}{\partial \eta}\right)^{-1}.\frac{1}{\left(\eta_i^{(n-1)} - \eta_k^{(n)}\right)} = 0 \qquad (15)$$

In the same way, by using (13) and (10), we have also :

$$\sum_{k=1}^{n}\left(\frac{\partial G^{(n)}\left(\eta_k^{(n)}\right)}{\partial \eta}\right)^{-1}.\frac{1}{\left(\eta_i^{(n-1)} - \eta_k^{(n)}\right)} = 0 \qquad (16)$$

4.3 New expression of reflection coefficient

By identification of Φ_n with two different approaches :

$$\Phi_n = \begin{bmatrix}\alpha_{n-1} & \alpha_{n-1}.A_{n-1}^+ \\ \alpha_{n-1}.A_{n-1} & \Phi_{n-1} + \alpha_{n-1}.A_{n-1}.A_{n-1}^+\end{bmatrix} = \sum_{k=1}^{n}\eta_k^{(n)}.X_k^{(n)}.X_k^{(n-1)+}$$

we can express reflection coefficients in an other way :

$$\mu_{n-1} = a_{n-1}^{(n-1)} = \frac{\sum_{k=1}^{n}\eta_k^{(n)}.X_{k,n}^{(n)}.X_{k,1}^{(n)*}}{\alpha_{n-1}} \quad \text{and} \quad \alpha_{n-1} = \sum_{k=1}^{n}\eta_k^{(n)}.\left|X_{k,1}^{(n)}\right|^2$$

$$\mu_{n-1} = \frac{2.\sum_{k=1}^{n}\eta_k^{(n)}.X_{k,n}^{(n)}.X_{k,1}^{(n)*}}{\sum_{k=1}^{n}\eta_k^{(n)}.\left[\left|X_{k,n}^{(n)}\right|^2 + \left|X_{k,1}^{(n)}\right|^2\right]} \underset{\eta_k^{(n)} \underset{n>M}{=} \alpha_n}{\approx} \underset{k>M}{\text{COV}}\left[X_{k,n}^{(n)}, X_{k,1}^{(n)}\right] \qquad (17)$$

5. RECURSIVE EIGENDECOMPOSITION VIA REFLECTION COEFFICIENT

5.1 Notations

$$\xi_k^{(n)} = A_n^{+}.\left(\frac{X_k^{(n)}}{X_{k,n}^{(n)}}\right) \quad \text{and} \quad \gamma_k^{(n)} = A_n^{(-)+}.\left(\frac{X_k^{(n)}}{X_{k,1}^{(n)}}\right) \tag{18}$$

$$f_k^{(n)} = \left[\frac{\partial F^{(n)}\left(\eta_k^{(n)}\right)}{\partial \eta}\right]^{-1} \quad \text{and} \quad g_k^{(n)} = \left[\frac{\partial G^{(n)}\left(\eta_k^{(n)}\right)}{\partial \eta}\right]^{-1} \tag{19}$$

$$\phi_k^{(n)} = \varphi_k^{(n)*} = X_{k,n}^{(n)}.X_{k,1}^{(n)*}$$

$$\sigma_i^{(n-1)}\left(\eta_k^{(n)}\right) = \frac{f_i^{(n-1)}.\gamma_i^{(n-1)*}}{\left(\eta_i^{(n-1)} - \eta_k^{(n)}\right)} \quad \text{and} \quad \rho_i^{(n-1)}\left(\eta_k^{(n)}\right) = \frac{g_i^{(n-1)}.\xi_i^{(n-1)*}}{\left(\eta_i^{(n-1)} - \eta_k^{(n)}\right)} \tag{20}$$

5.2 Eigenvalues and eigenvectors

By using previous notations, we have developed the following equations. Eigenvalues are roots of :

$$\begin{cases} F^{(n)}\left(\eta_k^{(n)}\right) = \eta_k^{(n)} - \alpha_{n-1} + \left(1 - |\mu_n|^2\right)\alpha_{n-1}^2.\sum_{i=1}^{n-1}\frac{\rho_i^{(n-1)}.\xi_i^{(n-1)}}{\eta_i^{(n-1)}} = 0 \\ G^{(n)}\left(\eta_k^{(n)}\right) = \eta_k^{(n)} - \alpha_{n-1} + \left(1 - |\mu_n|^2\right)\alpha_{n-1}^2.\sum_{i=1}^{n-1}\frac{\sigma_i^{(n-1)}.\gamma_i^{(n-1)}}{\eta_i^{(n-1)}} = 0 \end{cases} \tag{21}$$

where :

$$\begin{cases} f_k^{(n)} = \left[1 + \left(1 - |\mu_{n-1}|^2\right).\alpha_{n-1}^2.\sum_{i=1}^{n-1}\frac{\rho_i^{(n-1)}.\xi_i^{(n-1)}}{\left(\eta_i^{(n-1)} - \eta_k^{(n)}\right)}\right]^{-1} \\ g_k^{(n)} = \left[1 + \left(1 - |\mu_{n-1}|^2\right).\alpha_{n-1}^2.\sum_{i=1}^{n-1}\frac{\sigma_i^{(n-1)}.\gamma_i^{(n-1)}}{\left(\eta_i^{(n-1)} - \eta_k^{(n)}\right)}\right]^{-1} \end{cases} \tag{22}$$

and eigenvectors are provided by :

$$\begin{cases} \dfrac{X_k^{(n)}}{X_{k,1}^{(n)}} = \begin{bmatrix} 1 \\ -\eta_k^{(n)}.\left(1 - |\mu_{n-1}|^2\right).\alpha_{n-1}.\left[\dfrac{X_1^{(n-1)}}{X_{1,n-1}^{(n-1)}} \cdots \dfrac{X_{n-1}^{(n-1)}}{X_{n-1,n-1}^{(n-1)}}\right].\begin{bmatrix} \dfrac{\rho_1^{(n-1)}}{\eta_1^{(n-1)}} \\ \vdots \\ \dfrac{\rho_{n-1}^{(n-1)}}{\eta_{n-1}^{(n-1)}} \end{bmatrix} \end{bmatrix} \\ \dfrac{X_k^{(n)}}{X_{k,n}^{(n)}} = \begin{bmatrix} 1 \\ -\eta_k^{(n)}.\left(1 - |\mu_{n-1}|^2\right).\alpha_{n-1}.\left[\dfrac{X_1^{(n-1)}}{X_{1,1}^{(n-1)}} \cdots \dfrac{X_{n-1}^{(n-1)}}{X_{n-1,1}^{(n-1)}}\right].\begin{bmatrix} \dfrac{\sigma_1^{(n-1)}}{\eta_1^{(n-1)}} \\ \vdots \\ \dfrac{\sigma_{n-1}^{(n-1)}}{\eta_{n-1}^{(n-1)}} \end{bmatrix} \end{bmatrix} \end{cases} \tag{23}$$

5.3 Levinson Equation Utilization

Levinson equation allows to decrease computational complexity by introducing a reflection coefficient and to increase robustness by regularization. If we consider the following equation deduced from (1) :

$$T_{n-1} = \begin{bmatrix} 1 \\ A_{n-1} \end{bmatrix} = \begin{bmatrix} 1 \\ A_{n-2} \\ 0 \end{bmatrix} + \mu_{n-1}.\begin{bmatrix} 0 \\ A_{n-2}^{(-)} \\ 1 \end{bmatrix} = \begin{bmatrix} T_{n-2} \\ 0 \end{bmatrix} + \mu_{n-1}.\begin{bmatrix} 0 \\ T_{n-2}^{(-)} \end{bmatrix}$$

and equation (4), it provides a recursive equation about the following vectors product :

$$T_{n-1}^{+}.X_k^{(n)} = T_{n-2}^{+}.\overline{X}_k^{(n)} + \mu_{n-1}.T_{n-2}^{(-)+}.\underline{X}_k^{(n)} = \frac{\eta_k^{(n)}.X_{k,1}^{(n)}}{\alpha_{n-1}}$$

In the same way, if we use equation (9) and Levinson equation, we obtain this associated equation :

$$T_{n-1}^{(-)+}.X_k^{(n)} = T_{n-2}^{(-)+}.\underline{X}_k^{(n)} + \mu_{n-1}^{*}.T_{n-2}^{+}.\overline{X}_k^{(n)} = \frac{\eta_k^{(n)}.X_{k,n}^{(n)}}{\alpha_{n-1}}$$

With the previously defined notations, it leads to :

$$\begin{cases} \varphi_k^{(n)} = -\dfrac{\alpha_{n-1}.g_k^{(n-1)}}{\eta_k^{(n)}}.\dfrac{\left[\alpha_{n-1}.\sum_{i=1}^{n-1}\sigma_i^{(n-1)}\right]}{\left[1 + \mu_{n-1}.\alpha_{n-1}.\sum_{i=1}^{n-1}\rho_i^{(n-1)}\right]} \\ \phi_k^{(n)} = -\dfrac{\alpha_{n-1}.f_k^{(n-1)}}{\eta_k^{(n)}}.\dfrac{\left[\alpha_{n-1}.\sum_{i=1}^{n-1}\rho_i^{(n-1)}\right]}{\left[1 + \mu_{n-1}^{*}.\alpha_{n-1}.\sum_{i=1}^{n-1}\sigma_i^{(n-1)}\right]} \end{cases} \tag{24}$$

Levinson equation (1) also provides :

$$A_n = \begin{bmatrix} A_{n-1} \\ 0 \end{bmatrix} + \mu_n.\begin{bmatrix} A_{n-1}^{(-)} \\ 1 \end{bmatrix} = \begin{bmatrix} A_{n-1} \\ 0 \end{bmatrix} + \mu_n.T_{n-1}^{(-)}$$

$$A_n^{+}.X_k^{(n)} = A_{n-1}^{+}.\overline{X}_k^{(n)} + \mu_n.T_{n-1}^{(-)+}.X_k^{(n)}$$

In the same way, we obtain :

$$A_n^{(-)+}.X_k^{(n)} = A_{n-1}^{(-)+}.\underline{X}_k^{(n)} + \mu_n^{*}.T_{n-1}^{+}.X_k^{(n)}$$

By using equations (4,9) and (6,10), equations are reduced to :

$$\begin{cases} \xi_k^{(n)} = \dfrac{\eta_k^{(n)}}{\alpha_{n-1}}.\left[\mu_n - \alpha_{n-1}.\sum_{i=1}^{n-1}\dfrac{\phi_i^{(n-1)}.\xi_i^{(n-1)}.\gamma_i^{(n-1)*}}{\left(\eta_i^{(n-1)} - \eta_k^{(n)}\right)}\right] \\ \gamma_k^{(n)} = \dfrac{\eta_k^{(n)}}{\alpha_{n-1}}.\left[\mu_n^{*} - \alpha_{n-1}.\sum_{i=1}^{n-1}\dfrac{\varphi_i^{(n-1)}.\gamma_i^{(n-1)}.\xi_i^{(n-1)*}}{\left(\eta_i^{(n-1)} - \eta_k^{(n)}\right)}\right] \end{cases} \tag{25}$$

5.4 New Recursive Eigendecomposition

We have developed a new recursive eigendecompostion algorithm via reflection coefficient :

$$\mu_{n-1} = a_{n-1}^{(n-1)} = \frac{\sum_{k=1}^{n}\eta_k^{(n)}.X_{k,n}^{(n)}.X_{k,1}^{(n)*}}{\alpha_{n-1}} = \frac{\sum_{k=1}^{n}\eta_k^{(n)}.\phi_k^{(n)}}{\alpha_{n-1}} \tag{26}$$

This coefficient will be computed by an AR analysis.

6. AGO-ANTAGONISTIC REGULARIZATION

We have developed different approaches [1,2] to compute μ_n:

6.1 Maximum Entropy Approach : Classical Burg

$$f_m(n) = \sum_{k=0}^{m} a_k^{(m)}.x_{n-k} \;,\quad b_m(n) = \sum_{k=0}^{m} a_k^{(m)*}.x_{n-m+k} \quad \text{and} \quad a_0^{(m)} = 1$$

$$E^{(m)} = U^{(m)} \quad \text{with} \quad U^{(m)} = \frac{1}{2.(N-m)}\sum_{n=m+1}^{N}\left|f_m(n)\right|^2 + \left|b_m(n)\right|^2$$

$$\nabla_{\mu_m}U^{(m)} = \mu_m.G^{(m)} + D^{(m)*} = 0 \Rightarrow \mu_m = -\frac{D^{(m)*}}{G^{(m)}} \tag{27}$$

$$\text{with} \begin{cases} G^{(m)} = \dfrac{1}{N-m}\sum_{n=m+1}^{N}\left|f_{m-1}(n)\right|^2 + \left|b_{m-1}(n-1)\right|^2 \\ D^{(m)} = \dfrac{2}{N-m}\sum_{n=m+1}^{N}b_{m-1}(n-1).f_{m-1}^{*}(n) \end{cases}$$

6.2 Minimum Free Energy Approach : Regularized Burg

$$E^{(m)} = U^{(m)} + \sum_{k=0}^{1}\gamma_k M_k^{(m)} \text{with } M_k^{(m)} = \int_{-1/2}^{1/2}\left|\frac{d^k A^{(m)}(f)}{df^k}\right|^2 df$$

$$A^{(m)}(f) = \sum_{k=0}^{m} a_k^{(m)} e^{-j\omega k} = A^{(m-1)}(f) + \mu_m e^{-j\omega m} A^{(m-1)*}(f)$$

$$\textbf{let} \begin{cases} \mathbf{D_{reg}^{(m)} = D^{(m)} + \left[2.\sum_{k=1}^{m-1}\beta_k^{(m)}.a_k^{(m-1)}.a_{m-k}^{(m-1)}\right]^{*}} \\ \mathbf{G_{reg}^{(m)} = G^{(m)} + 2.\sum_{k=0}^{m-1}\beta_k^{(m)}.\left|a_k^{(m-1)}\right|^2} \end{cases} \tag{28}$$

$$\boldsymbol{\mu_m = -\frac{D_{reg}^{(m)*}}{G_{reg}^{(m)}}} \quad \textbf{and} \quad \boldsymbol{\beta_k^{(m)} = \gamma_0 + \gamma_1.(2.\pi)^2.(k-m)^2}$$

6.3 Minimum Free Enthalpy Approach : Ago-antagonistic Burg

$$E^{(m)} = U^{(m)} + \sum_{k=0}^{1}\gamma_k.M_k^{(m)} + \delta.\mathrm{Ln}\left[1 - |\mu_m|^2\right]$$

with $\mu_m = (-1)^m . \prod_{i=1}^{m} z_i^{(m)}$ and $\nabla_{\mu_m} Ln\left[1-|\mu_m|^2\right] = \frac{-2.\mu_m}{1-|\mu_m|^2}$

$$D_{reg}^{(m)*} + \mu_m.G_{reg}^{(m)} = \frac{2.\delta.\mu_m}{1-|\mu_m|^2} \quad \text{but} \quad \mu_m.D_{reg}^{(m)} \in \Re \qquad (29)$$

we set $\xi_m = \frac{\mu_m.D_{reg}^{(m)}}{\left|D_{reg}^{(m)}\right|}$, $|\xi_m| < 1$ root of

$$\left(1-\xi_m^2\right).\left(\xi_m.G_{reg}^{(m)} + \left|D_{reg}^{(m)}\right|\right) = 2.\delta.\xi_m \quad \text{and} \quad \mu_m = \frac{\xi_m.D_{reg}^{(m)*}}{\left|D_{reg}^{(m)}\right|}$$

δ is optimal when the straight line of the right term is tangential to 3rd order polynomial of the left term :

$$\begin{cases} Q(\xi_m) = \left(1-\xi_m^2\right).\left(\xi_m.G_{reg}^{(m)} + \left|D_{reg}^{(m)}\right|\right) = 2.\delta_{opt}.\xi_m \\ \frac{dQ(\xi_m)}{d\xi_m} = 2.\delta_{opt} \end{cases} \qquad (30)$$

Final result is computed by a substitution method [2].

7. RESULTS

7.1 Recursive Eigendecomposition

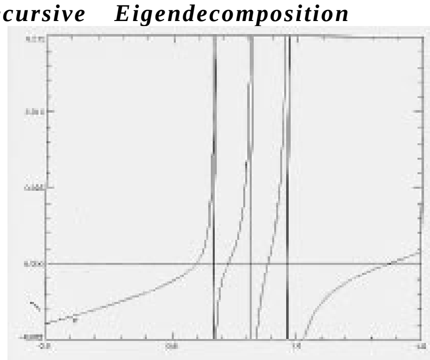

Fig.1 : $F^{(4)}(\eta)$ for 8 complex samples

7.2 Classical and Regularized Burg Spectrum

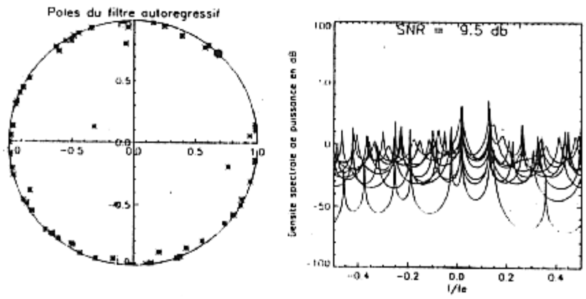

Fig. 2.1 ME Spectrum and poles with 2 eigenfrequencies with 10 complex samples

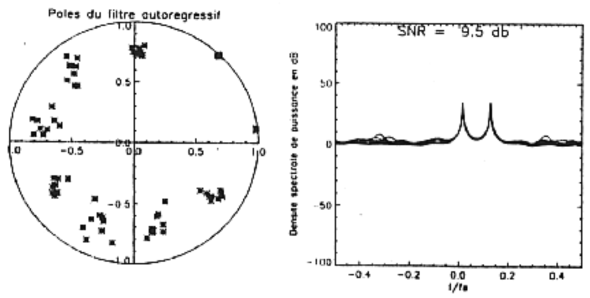

Fig. 2.2 Regularized Spectrum and poles

7.3 Ago-antagonistic Burg Spectrum

Time-doppler spectrum analysis of 8 complex radar samples from an helicopter data records :

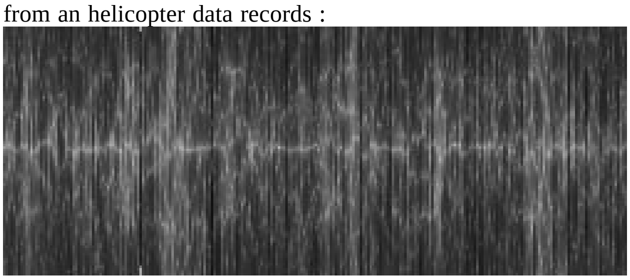

Fig.3.1 Classical time-doppler Burg Spectrum

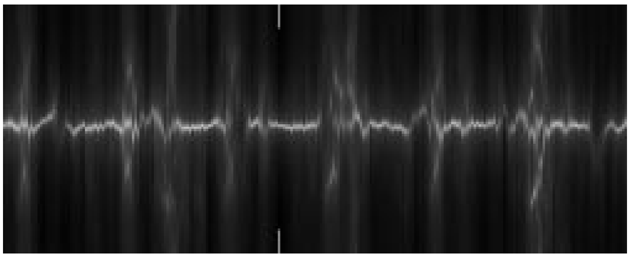

Fig.3.2 Regularized time-doppler Spectrum

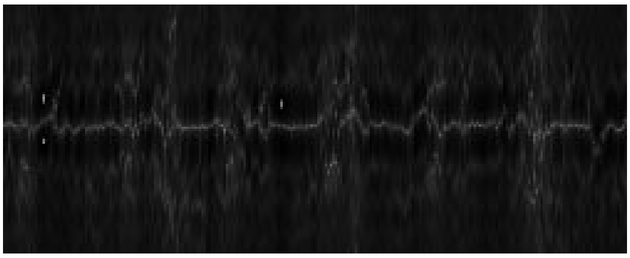

Fig.3.3 Ago-antagonistic time-doppler Spectrum

Ago-antagonism avoids smoothing effects of over-regularization methods and allows to restore some fine details by increasing spectrum resolution.

8. CONCLUSION

We have developed a new algorithm that finds the complete eigenspace decomposition of successively larger Hermitian Tœplitz matrix. Computation and robustness performances are provided by the ago-antagonistic reflection coefficient.

9. REFERENCES

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