SPECTRAL ESTIMATION OF A GAUSSIAN SIGNAL SAMPLED WITH JITTER

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ABSTRACT

This communication tackles the problem of a Gaussian band-limited continuous signal with unknown characteristics sampled with jitter. Under this weak assumption, we demonstrate a relation linking the power spectral density of the continuous signal to the second and fourth order statistics of the measured samples. A fundamental point is that this relation does not require the knowledge of the jitter characteristics. This result can be exploited for the derivation of spectral estimation algorithms when the jitter is unknown or jitter detection tests when the sampled signal is unknown. A simulation of spectral estimation confirms the validity of the result.

1. INTRODUCTION

Sampling jitter arises in many applications. They include spectral estimation, [7] or source localization, [8]. The study of sampling error has recently received a growing interest. In [1, 3, 10], the authors address the estimation of the jitter variance for a signal with known or specific spectral properties.

This communication, tackles the problem of a Gaussian signal with unknown characteristics sampled with jitter. Under this weak assumption, we show that the power spectral density (PSD) of the continuous signal can be recovered from a combined use of the sampled signal second and fourth order statistics. This result is validated by a computer experiment where the PSD of a signal sampled with jitter is recovered.

2. PROBLEM FORMULATION AND NOTATIONS

A real stationary Gaussian signal $x(t)$ having PSD $S(\omega)$ and autocorrelation $c(\tau)$ is considered. $x(t)$ is assumed to be a low-pass signal with $S(\omega) = 0$ for $|\omega| > \pi/h$. $x(t)$ is sampled according to Shannon theorem by a sampler that exhibits jitter. The sampling instants are $t_n = nh + \gamma_n$ where γ_n is a zero-mean iid sequence with characteristic function $\Phi(\omega) = \mathsf{E}[\exp(j\omega\gamma_n)]$. The parameter h is fixed to 1 without loss of generality and the samples at the output of the sampler are denoted $x_n = x(t_n)$. The functions $S(\omega)$ and $\Phi(\omega)$ are unknown. The paper addresses the problem of estimating the PSD $S(\omega)$ using the observations x_n.

3. THE SECOND AND FOURTH ORDER STATISTICS OF X_N

3.1. Power Spectral Density of x_n

Using the second order stationarity property of $x(t)$ and conditional expectations leads to:

$$c_h(n) = \mathsf{E}[x_n x_{m+n}] = \mathsf{E}[c(t_{m+n} - t_n)].$$

Replacing in this expression $c(\tau)$ by the inverse Fourier transform of $S(\omega)$ yields:

$$\forall m \neq 0,\, c_h(m) = \int_{-\pi}^{\pi} |\Phi(u)|^2 S(u) e^{jum} du$$
$$c_h(0) = \int_{-\pi}^{\pi} S(u) du = c(0).$$

$c_h(0)$ can be expressed as :

$$c_h(0) = \int_{-\pi}^{\pi} |\Phi(u)|^2 S(u) du + \int_{-\pi}^{\pi} S(u)(1 - |\Phi(u)|^2) du.$$

The PSD of x_n is then given by:

$$S_h(u) = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} c_h(n) e^{-jun}$$
$$= |\Phi u)|^2 S(u) + \frac{1}{2\pi} \int_{-\pi}^{\pi} S(u)(1 - |\Phi(u)|^2) du.$$

The modulus of $\Phi(u)$ being upper bounded by 1, the second integral is positive. Consequently, the effect of the jitter can be interpreted at the second order as a filtering process followed by an additive noise. Unfortunately, this expression does not allow to achieve our initial goal. For this sake, a new process $\tilde{x}_n$ with autocorrelation:

$$\forall m, \quad \tilde{c}_h(m) = \int_{-\pi}^{\pi} |\Phi(u)|^2 S(u) e^{jum} du,$$

will be considered. The PSD of $\tilde{x}_n$ satisfies:

$$\tilde{S}_h(u) = |\Phi(u)|^2 S(u). \tag{1}$$

The fact that $\tilde{S}_h(u)$ is positive confirms that $\tilde{c}_h(n)$ is a valid autocorrelation sequence. Moreover, this expression shows the equivalence between the problem of estimation of $|\Phi(u)|$ and $S(u)$ on $[-\pi, \pi]$. For this reason, we will focus on the sequel on the PSD estimation. Once this last is obtained, (1) can be used to estimate parameters of the jitter. For example, if the jitter variance σ_j^2 is required, substituting in (1) $|\Phi(u)|^2$ by its second order expansion, $1 - u^2\sigma_j^2$, for $u^2\sigma_j^2 \ll 1$, gives:

$$u^2\sigma_j^2 \approx \log(S(u)/\tilde{S}_h(u)).$$

The first step to obtain an estimate of $\tilde{S}_h(u)$ is the estimation of $c_h(n)$ for $n \neq 0$ directly from the x_n. The following step is the estimation of $\tilde{c}_h(0)$. A solution to obtain this quantity is to use two samplers with independent jitters γ_n et γ'_n, [9, 2]. If t'_n denotes the instants of sampling of the second sampler:

$$\mathsf{E}[x(t_n)x(t'_n)] = \int_{-\infty}^{+\infty} c(\tau)p(\tau)d\tau,$$

where $p(\tau)$ is the PDF of $\gamma_n - \gamma'_n$ and consequently $\mathsf{E}[x(t_n)x(t'_n)] = \tilde{c}_h(0)$. Note that, instead of using a DFT of the $\tilde{c}_h(m)$, an autoregressive model can be fitted to the autocorrelations of $\tilde{x}_n$ to obtain the estimated spectrum.

3.2. Trispectrum of x_n

Denote as $c_h(k,l,m)$ the fourth order cumulants of x_n. The derivation of $c_h(0,1,1)$ obtained in [1] can be easily generalized to every cumulant. This leads to the following result:

Theorem 1 *The only non-zero fourth order cumulants of x_n are $c_h(l-p,l,l)$ with $p \neq 0$. Their expression is:*
- *for $p \neq 0$, $l \neq p$:*

$$c_h(l-p,l,l) = 2\iint_{-\pi}^{\pi} \Xi(\alpha,\beta)S(\alpha)S(\beta)e^{j(\alpha l+\beta p)}d\alpha d\beta,$$
$$\Xi(\alpha,\beta) = \Phi(\alpha+\beta)\Phi^*(\alpha)\Phi^*(\beta) - |\Phi(\alpha)\Phi(\beta)|^2.$$

- *for $l \neq 0$:*

$$c_h(0,l,l) = 2\iint_{-\pi}^{\pi} \Upsilon(\alpha,\beta)S(\alpha)S(\beta)e^{j(\alpha+\beta)l}d\alpha d\beta,$$
$$\Upsilon(\alpha,\beta) = |\Phi(\alpha+\beta)|^2 - |\Phi(\alpha)\Phi(\beta)|^2.$$

Proof: $cum(x(t_n), x(t_{n+k}), x(t_{n+l}), x(t_{n+m}))$ is first replaced by its definition using the fourth and second order moments of $x(t_n)$. In this expression, each expectation is substituted by a conditionnal expectation. The fourth order moment of the Gaussian signal $x(t)$ is then replaced by its expression as a function of its second order moments. This gives:

$$\begin{aligned} c_h(k,l,m) = &\mathsf{E}[c(t_{n+k}-t_n)c(t_{n+m}-t_{n+l})] \\ &+ \mathsf{E}[c(t_{n+l}-t_n)c(t_{n+m}-t_{n+k})] \\ &+ \mathsf{E}[c(t_{n+m}-t_n)c(t_{n+l}-t_{n+k})] \\ &- \mathsf{E}[c(t_{n+k}-t_n)]\mathsf{E}[c(t_{n+m}-t_{n+l})] \\ &- \mathsf{E}[c(t_{n+l}-t_n)]\mathsf{E}[c(t_{n+m}-t_{n+k})] \\ &- \mathsf{E}[c(t_{n+m}-t_n)]\mathsf{E}[c(t_{n+l}-t_{n+k})]. \end{aligned}$$

The analyze of this expression, using the independence of the jitter, shows that the only non zero terms occur for $l = m$ and $k \neq l$. In this case, the cumulant reduces to:

$$\begin{aligned} c_h(k,l,l) = &2\mathsf{E}[c(t_{n+l}-t_n)c(t_{n+l}-t_{n+k})] \\ &- 2\mathsf{E}[c(t_{n+l}-t_n)]\mathsf{E}[c(t_{n+l}-t_{n+k})]. \end{aligned}$$

Replacing in this expression $c(\tau)$ by the inverse Fourier transform of $S(\omega)$ and using again the independence of the jitter terminates the proof. □

The trispectrum of x_n, $T_h(u_1,u_2,u_3)$, defined on the usual domain $|u_1|, |u_2|, |u_3| < \pi$, $|u_1+u_2+u_3| < \pi$ equals:

$$T_h(u_1,u_2,u_3) = \frac{1}{(2\pi)^3}\sum_{l,p;p\neq 0} c_h(l-p,l,l) e^{-j((u_1+u_2+u_3)l-u_1p)}.$$

If $u = -u_1$ and $v = u_1+u_2+u_3$, a new function $\Gamma_h(u,v)$ can be defined for $|u|, |v| < \pi$:

$$\begin{aligned} \Gamma_h(u,v) &= \pi T_h(u_1,u_2,u_3) \\ &= \frac{1}{8\pi^2}\sum_{l,p;p\neq 0} c_h(l-p,l,l)e^{-j(vl+up)}. \end{aligned} \tag{2}$$

The next step is the substitution of the cumulants by their expression given by theorem 1. This suggests a decomposition of $\Gamma_h(u,v)$ in two terms:

$$\Gamma_h(u,v) = \Gamma_1(u,v) + \Gamma_2(u,v),$$

where in $\Gamma_1(u,v)$ the sum ranges over $l \neq p$ and in $\Gamma_2(u,v)$ the sum ranges over $l = p$. The evaluation of these terms using the Poisson sum formula gives:

$$\Gamma_1(u,v) = \sum_{i=1}^{4} \Gamma_1^i(u,v), \quad \Gamma_2(u,v) = \sum_{i=1}^{4} \Gamma_2^i(u,v),$$

where:

$$\Gamma_1^1(u,v) = \Xi(u,v)S(u)S(v)$$
$$\Gamma_1^2(u,v) = \frac{-1}{2\pi}\int_{-\pi}^{\pi} \Xi(\alpha, u+v-\alpha)S(\alpha)S(u+v-\alpha)d\alpha$$
$$\Gamma_1^3(u,v) = \frac{1}{(2\pi)^2}\iint_{-\pi}^{\pi} \Xi(\alpha,\beta)S(\alpha)S(\beta)d\alpha d\beta$$
$$\Gamma_1^4(u,v) = \frac{-1}{2\pi}\int_{-\pi}^{\pi} \Xi(v,\beta)S(v)S(\beta)d\beta$$
$$\Gamma_2^1(u,v) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Upsilon(\alpha, u+v-\alpha)S(\alpha)S(u+v-\alpha)d\alpha$$
$$\Gamma_2^2(u,v) = \Gamma_{2,2}(u, v+2\pi)$$
$$\Gamma_2^3(u,v) = \Gamma_{2,2}(u, v-2\pi)$$
$$\Gamma_2^4(u,v) = \frac{1}{(2\pi)^2}\iint_{-\pi}^{\pi} \Upsilon(\alpha,\beta)S(\alpha)S(\beta)d\alpha d\beta.$$

4. EXPRESSION OF $S(\omega)$

The problem is to obtain from $\Gamma_h(u,v)$ and $\tilde{S}_h(u)$ an expression of $S(\omega)$ that does no depend on $\Phi(.)$. A solution of this problem is given by the following theorem that is the major contribution of this communication.

Theorem 2 *For a symmetric zero mean jitter, we have:*

$$S(\omega)\tilde{S}_h(\omega) = \tilde{S}_h(\omega)^2 + 2\int_0^{\omega} \left.\frac{\partial \Gamma_h(u,v)}{\partial u}\right|_{v=-u} du. \quad (3)$$

Proof: Each $\Gamma_p^q(u,v)$ is first differentiate with respect to u and v replaced by $-u$. The following properties verified in part by the characteristic function of a symmetric and zero mean random variable:

$$\Phi(u) = \Phi(-u) = \Phi(u)^*, \Phi'(0) = 0,$$
$$\Phi'(-u) = -\Phi'(u), \Phi(0) = 1, S(-u) = S(u).$$

are then used to simplify the expressions. After some calculation we obtain:

1. $\partial\Gamma_1^2(u,v)/\partial u\big|_{v=-u}$ and $\partial\Gamma_2^1(u,v)/\partial u\big|_{v=-u}$ involve the integrals of odd functions on $[-\pi,\pi]$ and consequently equal zero.
2. $\partial\Gamma_2^2(u,v)/\partial u\big|_{v=-u}$ and $\partial\Gamma_2^3(u,v)/\partial u\big|_{v=-u}$ involves the integral $S(\alpha\pm 2\pi)$ that equals zero for $|\alpha| \leq \pi$.
3. The differentiation of $\Gamma_1^3(u,v)$, $\Gamma_1^4(u,v)$, $\Gamma_2^3(u,v)$ with respect to u is obviously zero.

Consequently, the only non zero term corresponds to $\partial\Gamma_{1,1}(u,v)/\partial u\big|_{v=-u}$ and equals:

$$\left.\frac{\partial \Gamma_h(u,v)}{\partial u}\right|_{v=-u} = (\Phi'(u) - 2\Phi(u)^2\Phi'(u))\Phi(u)S(u)^2 + (1-\Phi(u)^2)\Phi(u)^2 S'(u)S(u). \quad (4)$$

The differentiation of (1) with respect to u gives:

$$\tilde{S}_h'(u) = \Phi(u)(2\Phi'(u)S(u) + \Phi(u)S'(u)). \quad (5)$$

The substitution of (1) and (5) in (4) leads to:

$$2\left.\frac{\partial \Gamma_h(u,v)}{\partial u}\right|_{v=-u} = (S(u)\tilde{S}_h(u))' - (\tilde{S}_h(u)^2)'.$$

This equation can now be integrated on $[0,\omega]$. Equation (1) implying $\tilde{S}_h(0) = S(0)$, we finally obtain (3). □

5. ESTIMATION OF THE CORRECTIVE TERM

The last step is to derive an expression of the integral that appears in (3) as a function of the fourth order cumulants of x_n. Differentiation of (2) gives:

$$\frac{\partial \Gamma_h(u,v)}{\partial u} = \frac{-j}{8\pi^2}\sum_{l,p;p\neq 0} p c_h(l-p,l,l)e^{-j(vl+up)}.$$

The integration of this equality where v has been substituted with $-u$ is:

$$\int_0^{\omega} \left.\frac{\partial \Gamma_h(u,v)}{\partial u}\right|_{v=-u} du = \frac{-j\omega}{8\pi^2}\sum_{l,p;p\neq 0} p c_h(l-p,l,l)\,\mathrm{sin}_c((p-l)\omega/2)e^{-j(p-l)\omega/2}.$$

Finaly, the two PSD in (3) being real valued, the previous expression can be simplified to:

$$\int_0^{\omega} \left.\frac{\partial \Gamma_h(u,v)}{\partial u}\right|_{v=-u} du = \frac{\omega}{8\pi^2} \sum_k \mathrm{sin}_c(k\omega/2)\sin(k\omega/2)\sum_l (l-k)c_h(k,l,l). \quad (6)$$

A question that arises immediately is the convergence of the previous summations. It can be easily verified that if x_n is fourth order mixing in the sense defined in [5, p. 26], the summation over l converges. We will assume that the convergence of the summation over k is allowed by the decaying of the $\mathrm{sin}_c(\cdot)$.

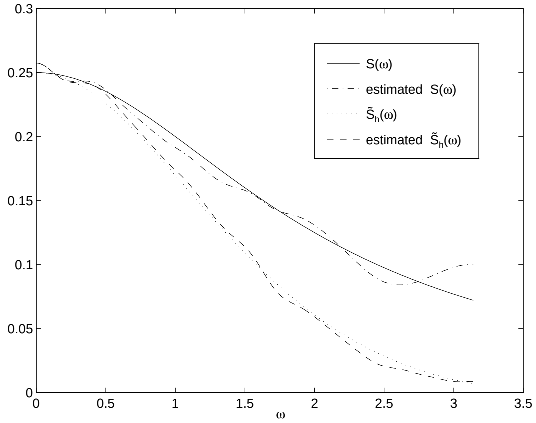

Figure 1: Estimation of $S(\omega)$.

6. SIMULATIONS AND CONCLUSIONS

To validate the previous results, (3) has been applied to PSD estimation. Neverthless, the expression of $S(\omega)$ involving a division by $\tilde{S}_h(\omega)$, (3), cannot be directly usable.

For this reason, we derive a slightly different estimation scheme. If we assume that $S(\omega)$ and $\tilde{S}_h(\omega)$ are of the same order of magnitude, which is verified for ω near 0, see (1), $\tilde{S}_h(\omega)$ can be substituted by $S(\omega)$ in the left side of (3). This leads to the following approximation of $S(\omega)$:

$$S(\omega)^2 \approx \tilde{S}_h(\omega)^2 + 2\int_0^{\omega} \left.\frac{\partial \Gamma_h(u,v)}{\partial u}\right|_{v=-u} du.$$

For the following experiment, the signal $x(t)$ is the Gaussian stationary process, [6]:

$$dx = -\alpha x dt + dv,$$

where: $\alpha > 0$, $v(t)$ is a Wiener process with variance parameter 2π and $x(0)$ a zero mean Gaussian variable with variance π/α. The PSD of this process is $S(\omega) = 1/(\alpha^2 + \omega^2)$. The discretisation of $x(t)$ at the instants t_n is, [4] :

$$x_{n+1} = e^{-\alpha(t_{n+1}-t_n)} x_n + (1 - e^{-2\alpha(t_{n+1}-t_n)})^{1/2} e_{n+1},$$

where e_n is iid Gaussian with variance π/α. As in [1], the jitter is a binary process taking the values $\{-0.4, 0.4\}$ and $\alpha = 2$. A realization of $N = 10^5$ samples have been drawn. From this process $\tilde{c}_h(m)$, $m = 0 \ldots 15$ and $c_h(k,l,l)$, $|k|, |l| \leq 6$, $k \neq l$, have been estimated. The cumulants have been computed by averaging the estimates obtained on segments of $N/30$ samples with 10% of overlap. Figure 1 shows the theoretical PSD $S(\omega)$, $\tilde{S}_h(\omega)$ and their estimated versions. These results clearly prove the validity of the mathematical derivations. The oscillation in the estimated PSD can be interpretated by both the truncation in the sumation of (6) and the fact that the process $x(t)$ for the simulation is not really low-pass and some aliasing is present. The use of (3) for jitter detection is currently under investigation.

7. REFERENCES

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