ALGORITHM DESIGN FOR STRUCTURED SYSTEMS: APPLICATION TO POLE PLACEMENT

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ABSTRACT

Numerical algorithms for signal processing and control are quite often constructed by intuition. When the system to be designed contains algebraic or other invariants, then these constraints can be exploited to find appropriate transformations. The transformations in system theory are usually Lie groups. One has to find Lie groups which are consistent with the invariants.

We show how this point of view can be applied to construct pole placement algorithms for symmetric and skew-symmetric realizations.

However, Lie group theory only reveals the appropriate transformations but is not able to reduce the design process to a trivial task. The problem discussed here also shows this limitation.

1. INTRODUCTION

System design is typically an iterative or recursive process starting from an initial design and application of transformations. In many cases this process is constrained by conditions on the systems parameters. They should be satisfied in all steps, i.e., by all transformations. State space transformations for linear systems are a well known example for this procedure. Nonsingular transformations of the state vector leave the transfer function invariant while changing the internal structure. Other examples from signal processing and systems theory are listed in the table below.

In this paper we consider linear time-invariant systems. Their parameters constitue a manifold. In this case Lie groups provide a powerful class of transformations. The structure comes in from physical constraints, e.g., reciprocity or losslessness. We take pole placement as an example because it is of interest in a variety of applications in linear systems theory. However, our way of thinking is appropriate for other problems as well.

Task	Constraint	Transformation
Eigenvalue dec.	eigenvalues	sim. trans.
Signature	signs of EVs	cong. trans.
lossless synth.	energy	appr. group act.
Feedback	given poles	this paper

Table 1: SP tasks, constraints and transformations

Issues of solvability, discussed in control literature over the last decades, are beyond the scope of this paper and will be published elsewhere.

2. INVARIANCE PRINCIPLE

Group theory proves to be indispensible for a systematic system design of algorithms. In fact, if the transformations applied in the algorithm are elements of a group, one can use rather powerful tools from group theory to investigate the behavior of the algorithm. Primarily, we are interested in the case of a group action on a manifold.

Structural properties of a linear system can be considered in this framework if it is possible to express the structural constraints as invariants under group actions. First we give a definition of what invariance means in this respect.

Definition 2.1 (Invariance [8]) *Let G be a transformation group acting on a manifold $\mathcal{M}$. An invariant of G is a real valued function $I : \mathcal{M} \to \mathbb{R}$, which satisfies*

$$I(g \cdot x) = I(x) \quad (\forall g \in G). \tag{1}$$

Given one or more invariants I and the action of the transformation group on $\mathcal{M}$ in terms of a mapping, groups G satisfying (1) have to be found. The group action is simply how the rules of transformations are applied to the system parameters as long as groups axioms are satisfied.

Example: For eigenvalues computation the group action is defined as $\boldsymbol{A} \mapsto \boldsymbol{TAT}^{-1}$ ($\boldsymbol{T} \in \mathrm{GL}(n)$). This group action leaves the eigenvalues invariant.

Remark: Continuous transformation groups leaving dynamical systems invariant were already considered S. Lie [6] and F. Klein. Here we apply their ideas to algebraic invariants.

Remark: Invariance priciples were applied in network synthesis for lossless systems. This is a special case of the general principle defined above, which goes beyond system theory.

3. PROBLEM DESCRIPTION

Assigning poles to linear time-invariant systems by feedback amounts to the solution of a set of nonlinear equations. Recently, pole placement problems were attacked by embedding plant and compensator in a projective space and using Plücker coodinates. This allows the application of optimization methods on Grassmannians as done by Helmke/Hüper [5] because the problem is reduced to a symmetric eigenvalue problem. This can be solved, e.g., using Jacobi-type methods.

We consider pole placement for symmetric (skew-symmetric) realizations, that is the realization matrix $\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} \in \mathbb{R}^{(m+n)\times(m+n)}$ of the linear time-invariant system becomes symmetric (skew-symmetric) and so are the transfer function and the feedback.

Such compensators for state space realizations were considered by Helmke and Mahony with gradient flow techniques [4], [7].

In this paper we show how to express structural constraints in terms of invariants and compute the admissible transformations according to Definition 2.1.

Let a linear time-invariant system $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ of McMillan degree n with m inputs and outputs, respectively, and a set of n real (imaginary) eigenvalues $\mathcal{L} = \{s_1, s_2, \ldots, s_n\}$ be given. Compute a feedback compensator $\boldsymbol{K}$ such that the eigenvalues of the closed-loop system match $\mathcal{L}$, that is,

$$\det(s_i \boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{BKB}^{\mathrm{T}})) = 0, \quad i = 1, \ldots, n \qquad (2)$$

and $\boldsymbol{K}$ is symmetric (skew-symmetric).

This condition can also be written as an intersection of linear spaces

$$\dim \left[\operatorname{colspan} \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{K} \end{bmatrix} \cap \operatorname{colspan} \begin{bmatrix} \boldsymbol{N}(s_i) \\ \boldsymbol{D}(s_i) \end{bmatrix} \right] \geq 1 \qquad i = 1, \ldots, n \qquad (3)$$

with transfer function $\boldsymbol{H} = \boldsymbol{ND}^{-1}$ and the special restriction to a symmetric (skew-symmetric) feedback $\boldsymbol{K}$.

4. GROUP ACTIONS

Now we express the structural constraints as invariants of group actions and determine the groups to be applied.

4.1. SYMPLECTIC STRUCTURE – SYMMETRIC FEEDBACK

Define a linear symplectic structure on $\mathbb{R}^{2m}$ by

$$\langle a, b \rangle = -\langle b, a \rangle = \sum_{i,i=1}^{2m} \omega_{ij} a_i b_j \quad \omega_{ij} = -\omega_{ji} \qquad (4)$$

and $\langle a, b \rangle = 0$ for $\forall a \in \mathbb{R}^{2m}$ if and only if $b = 0$. A subspace L of $\mathbb{R}^{2n}$ is called symplectic if the restriction to it of the symplectic structure is nondegenerate.

A k-plane W in the symplectic space is isotropic if it is skew-orthogonal to itself. For $k = m$ the plane is called Lagrangian [2], [1], [3]. It is the maximal isotropic subspace of a symplectic space.

The symmetry constraint of a system can be expressed in terms of (4) by

$$\begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{K} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} & \boldsymbol{I}_n \\ -\boldsymbol{I}_n & \end{bmatrix} \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{K} \end{bmatrix} = 0 \qquad (5)$$

for the graph of $\boldsymbol{K}$. These subspaces are called Lagrange Grassmannian. This means that the matrix $\boldsymbol{K}$ is symmetric. For the derivation of a proper algorithm it is crucial to characterize the group action, i.e., what matrix group leaves the space span $\begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{K} \end{bmatrix}$ isotropic.

$$\boldsymbol{Q} \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{K} \end{bmatrix} \in \operatorname{span} \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{K} \end{bmatrix}. \qquad (6)$$

It is easy to verify that symplectic matrices $\mathrm{Sp}(2m, \mathbb{R})$ work.

$$\begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{K} \end{bmatrix}^{\mathrm{T}} \underbrace{\boldsymbol{Q}^{\mathrm{T}} \begin{bmatrix} & \boldsymbol{I}_n \\ -\boldsymbol{I}_n & \end{bmatrix} \boldsymbol{Q}}_{\begin{bmatrix} & \boldsymbol{I}_n \\ -\boldsymbol{I}_n & \end{bmatrix}} \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{K} \end{bmatrix} = 0$$

$$\boldsymbol{Q} \in Sp(2m, R). \qquad (7)$$

Note that equation (5) is also called the condition for reciprocity of a linear (resitive) multiport.

We consider a linear system as a linear subspace. Clearly, every subspace can be described by a base subject to coordinate changes. It is convenient to work with an orthogonal base

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} I \\ K \end{bmatrix} (I + K^{\mathrm{T}} K)^{-1/2} \tag{8}$$

disregarding issues of numerical accuracy at this point. For the Lagrange Grassmannian $\begin{bmatrix} X \\ Y \end{bmatrix}$ (5) holds, subject to orthogonal coordinate changes by right multiplication. For the form $\begin{bmatrix} I \\ K \end{bmatrix}$, coordinate changes by the general linear group are sufficient. Note that right multiplication does not change the isotropy of the subspace. The projection matrix, parametrizing the Lagrange Grassmannian globally, is given by

$$P = \begin{bmatrix} X \\ Y \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}^{\mathrm{T}}. \tag{9}$$

Now the group action

$$\begin{bmatrix} X' \\ Y' \end{bmatrix} := Q \begin{bmatrix} X' \\ Y' \end{bmatrix} \tag{10}$$

has to be isotropic with respect to $\begin{bmatrix} & I_n \\ -I_n & \end{bmatrix}$ and has to preserve orthogonality in the Euclidean sense due to the orthogonality of (6). We obtain orthogonal symplectic matrices as admitted transformations

$$OSp(2m) := O(2m) \cap Sp(2m), \tag{11}$$

the maxiaml compact subgroup of the symplectic group. Such transformations have m^2 parameters. The Lie algebra $\mathfrak{osp}(2m)$ of these matrices is given by

$$\begin{aligned} \mathfrak{osp}(2m) \;=\; & \left\{ \begin{bmatrix} A & B \\ -B^{\mathrm{T}} & A \end{bmatrix}, A, B \in \mathbb{R}^{m \times m} | \right. \\ & \left. A = -A^{\mathrm{T}}, B = B^{\mathrm{T}} \right\}. \end{aligned} \tag{12}$$

As an aside, note that the additional orthogonality removes the noncompact part in $Sp(2m)$ and avoids difficulties in terms of function minimization along curves in $Sp(2m)$.

Remark: In [7], symmetric realizations written in state space form were considered. The similarity transformation was taken from $\mathrm{O}(n)$ because only such transformations leave the realization matrix symmetric.

4.2. LOSSLESS SYSTEMS

Assigning poles to a skew symmetric system is essentially a pole placement problem for Hamiltonian systems. Since the state matrix A is assumed to be skew, and this structure has to be preserved, only poles on or symmetric to the imaginary axis can be assigned.

The computation of the admitted transformation group goes along the same line as above but starting from the isotropy:

$$\begin{bmatrix} I \\ K \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} & I_n \\ I_n & \end{bmatrix} \begin{bmatrix} I \\ K \end{bmatrix} = 0. \tag{13}$$

This amounts to $K^{\mathrm{T}} = -K$. The Lie algebra for a matrix leaving (13) invariant and being orthogonal for the same reasons as above is given by:

$$\begin{aligned} \mathfrak{oso}(m, m) \;=\; & \left\{ \begin{bmatrix} A & B \\ -B^{\mathrm{T}} & A \end{bmatrix}, A, B \in \mathbb{R}^{m \times m} | \right. \\ & \left. A^{\mathrm{T}} = -A, B^{\mathrm{T}} = -B \right\}. \end{aligned} \tag{14}$$

This Lie algebra is of dimension $m^2 - m$.

5. ALGORITHM

The construction of a pole placement algorithm can be summarized by the following steps. Details for the problem will be given below.

Algorithm Design

1. Express the problem to be solved by the action of a Lie group
2. Write the structural constraint as isotropy condition (or invariance) of a group action and find the group which leaves the isotropy invariant.
3. Use this group to compute Plücker embedding or other strategies to solve the pole placement problem. The construction of an objective function for a quadratically convergent scheme will not be discussed here.

It was not possible to derive a quadratically convergent scheme for the objective functions in [5] and general K . This is to be expected for more the restricted transformation groups as well.

The intersection condition (2) can be satisfied by the vanishing determinants

$$\det \begin{bmatrix} I & N(s_i) \\ K & D(s_i) \end{bmatrix} = 0 \quad i = 1, \ldots n. \tag{15}$$

From these determinants an objective function is derived as

$$f(K) = \sum_{i=1}^{n} \det \begin{bmatrix} I & N(s_i) \\ K & D(s_i) \end{bmatrix}. \tag{16}$$

Details can be found in [5]. Following the exposition in [5] we consider the Plücker embedding of $Gr(2m,m)$ in the projective space

$$pl: \quad Gr(2m,m) \hookrightarrow \mathbb{P}(\wedge^m \mathbb{R}^{2m}). \tag{17}$$

This allows the objective function to be expressed as Rayleigh quotient

$$f(\boldsymbol{P}) = \sum_{i=0}^{n} pl(\boldsymbol{P})^{\mathrm{T}} \hat{\boldsymbol{Q}}_i pl(\boldsymbol{P}) \tag{18}$$

with $\boldsymbol{P}$ as given above and $\hat{\boldsymbol{Q}}_i$ the Plücker embedding of the orthogonal projector of $\begin{bmatrix} \boldsymbol{N}(s_i) \\ \boldsymbol{D}(s_i) \end{bmatrix}$

$$\hat{\boldsymbol{Q}}_i = pl(q_i) pl(q_i)^{\mathrm{T}}. \tag{19}$$

This is equivalent to the minimization of

$$h : pl(Gr(2m,m)) \to \mathbb{R},$$

thus

$$h(\boldsymbol{y}) = \mathrm{tr}\left[\boldsymbol{y}\boldsymbol{y}^{\mathrm{T}} \sum_{i=1}^{n} \hat{\boldsymbol{Q}}_i\right] \tag{20}$$

$\boldsymbol{y} = pl \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \end{bmatrix}$, initialized with $\boldsymbol{y}_0 = \begin{bmatrix} \boldsymbol{I}_m \\ 0_m \end{bmatrix}$. The iteration scheme applied to (20) is identical to algorithm (22) in [5] and is therefore not given here.

Example: The exponentiation of the exterior square of the standard representation of $\mathfrak{osp}(4)$ yields among others ($s = \sin$, $c = \cos$)

$$Q_2(\tau) = \begin{bmatrix} c(\tau)^2 & \frac{s(2\tau)}{2} & 0 & 0 & \frac{-s(2\tau)}{2} & -s(\tau)^2 \\ \frac{-s(2\tau)}{2} & c(\tau)^2 & 0 & 0 & s(\tau)^2 & \frac{-s(2\tau)}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{s(2\tau)}{2} & s(\tau)^2 & 0 & 0 & c(\tau)^2 & \frac{s(2\tau)}{2} \\ -s(\tau)^2 & \frac{s(2\tau)}{2} & 0 & 0 & \frac{-s(2\tau)}{2} & c(\tau)^2 \end{bmatrix}$$

$$Q_3(\tau) = \begin{bmatrix} c(t) & 0 & 0 & -s(\tau) & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & c(\tau) & 0 & 0 & s(\tau) \\ s(\tau) & 0 & 0 & c(\tau) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -s(\tau) & 0 & 0 & c(\tau) \end{bmatrix}$$

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It is worth mentioning that the chosen objective function suffers from slow convergence. This feature, however, does not result from the chosen transformations but rather the objective function.

6. CONCLUSIONS

In this paper we have shown how system invariants can be applied to find admissible transformation groups. This method is not restricted to pole placement. It is to be expected that this point of view can be used in many other applications as well. It is just a matter of patience to seek invariants. The group theory approach shows what transformations are allowed to leave the invariants fixed during the design process. We tacitly assume that this is the best route to the solution of a problem. Examples, e.g., similarity transformations of linear systems by non-singular matrices or eigenvalue computation of symmetric matrices by orthogonal matrices, support this assumption. Different algorithms exist for the symmetric and nonsymmetric case for the eigenvalues problem. The former also respects the symmetry under round-off errors while the latter does not obtain real eigenvalues when applied to a symmetric matrix. It seems not appropriate to apply arbitrary transformations at first and than to modify the desing in order to meet the constraints.

7. REFERENCES

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