

CONTINUOUS-TIME ENVELOPE-CONSTRAINED FILTER DESIGN VIA LAGUERRE FILTERS AND \mathcal{H}_∞ OPTIMIZATION METHODS*

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ABSTRACT

Envelope-constrained filtering is concerned with the design of a time-invariant filter to process a given input signal such that the noiseless output of the filter is guaranteed to lie within a prespecified output mask. In this paper, using Laguerre filters and \mathcal{H}_∞ optimization techniques, the continuous-time envelope-constrained filter design problem has been reformulated and solved as a constrained \mathcal{H}_∞ model-matching problem. To illustrate the effectiveness of the design method, a numerical example is presented which deals with the design of an equalization filter for a digital transmission channel.

1. INTRODUCTION

Envelope-constrained (EC) filtering is concerned with the design of a linear time invariant (LTI) filter $U(s)$ with impulse response $u(t)$ to process a given input pulse $s(t)$ which is corrupted by zero mean white noise $n(t)$, see Fig. 1(a). The noiseless output $\psi(t)$ is required to fit into a prespecified pulse shape envelope defined by the lower and upper boundaries $\epsilon^-(t)$ and $\epsilon^+(t)$ as shown in Fig. 1(b).

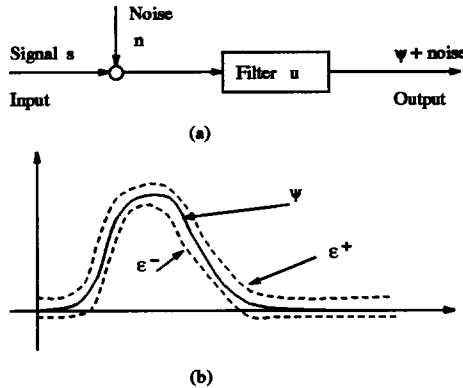


Figure 1. Envelope-constrained filtering problem: (a) Block diagram. (b) Pulse shape envelope

Previously, the optimal EC filter has been defined as the filter which minimizes the output noise power while satisfying the pulse shape constraints. It can be easily verified that the output noise power is proportional to the squared \mathcal{L}_2

norm of the filter. Hence the EC filtering problem, denoted as problem (P0), can be posed as

$$\min \|u\|_2^2 \quad \text{subject to} \quad \epsilon^-(t) \leq \psi(t) \leq \epsilon^+(t), \quad \forall t \in R_+$$

where $R_+ = [0, \infty)$ and

$$\|u\|_2^2 \triangleq \left(\int_0^\infty |u(t)|^2 dt \right)^{1/2}, \quad \psi(t) = \int_0^\infty s(\tau) u(t - \tau) d\tau.$$

Traditionally, problems of this type were often treated by minimizing the mean-square difference between ψ and some desired pulse shape. However, in many applications this "soft" least-squares approach is unsatisfactory because large narrow excursions from the desired shape occur and the norm of the filter can be large. It was argued in [3] that the EC filtering problem as defined above is more relevant than the "soft" least-squares approach in a variety of signal processing fields such as robust antenna and filter design [1], communication channel equalization [2], [6], and pulse compression in radar and sonar [8].

Although the EC filtering problem was initially posed in the continuous-time domain as a constrained \mathcal{L}_2 space optimization problem, only the discretized version has been solved using various approaches, see, e.g., [3], [12], [14]. Assume that both the input signal and the impulse response of the filter to be designed are time-limited \mathcal{L}_2 space functions, the solution to the discretized version EC filtering problem is an FIR filter. Although FIR filters are attractive due to their simplicity, they generally require a large number of taps. Furthermore, the number of taps needed in general turns out to be highly sensitive to sampling rate.

In this paper, we shall directly tackle the continuous time EC filtering problem by using continuous-time Laguerre series representation. Furthermore, instead of solving the constrained \mathcal{L}_2 minimization problem (P0), we shall seek to design an LTI filter $U(s)$ such that its \mathcal{H}_∞ norm, defined as $\|U\|_\infty = \sup_{\omega \in R} |U(j\omega)|$ (see, e.g., [4]), is minimized subject to the same constraints. We shall demonstrate that the use of Laguerre series representation and \mathcal{H}_∞ norm offers a more robust, low order alternative to FIR filters. Furthermore, the solution procedure does not involve discretization of either the filter or the input signal.

The use of orthogonal functions such as Laguerre filters for signal representation and filter synthesis is classical and can be traced back to the 1930's (see [5] for a summary of this early work), however, their application subject to time domain envelope constraints have not yet been addressed.

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The motivation for minimizing the \mathcal{H}_∞ norm of the filter is as follows: In the EC filtering problem (P0), the design objective is to minimize the output noise power while forcing the noiseless output response to fit into a prespecified output mask. The problem formulation depends on the assumption that the additive noise n is white with known constant spectral density. Even for a more general approach, the power spectral density function of the additive noise n has to be available for design. In practical applications, information about the noise's power spectral density is often limited. In this case, we may assume that the noise spectral density, denoted as $\Phi_N(\omega)$, is in a class \mathcal{B}_N of spectra bounded by a known upper bound. That is, $\mathcal{B}_N = \{\Phi_N : \Phi_N(\omega) \leq b_N(\omega)\}$. It is easy to verify that the output noise power due to the input noise n is given by

$$P_N = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_N(\omega) |U(j\omega)|^2 d\omega.$$

Assume that for any $\Phi_N \in \mathcal{B}_N$ its noise power satisfies $\|\Phi_N\| \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_N(\omega) d\omega \leq 1$ but otherwise unknown (Since \mathcal{B}_N is a bounded set, this can be achieved through scaling). It makes sense to design a filter $U(s)$ so that the output noise power P_N is minimized for the worst case input noise in \mathcal{B}_N . That is, we seek to solve the following minimax problem subject to the time domain constraint specified in problem (P0).

$$\min_U \max_{\|\Phi_N\| \leq 1} P_N = \min_U \max_{\|\Phi_N\| \leq 1} \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_N(\omega) |U(j\omega)|^2 d\omega.$$

It is known (see, e.g., [4]) that

$$\|U\|_\infty^2 = \max_{\|\Phi_N\| \leq 1} \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_N(\omega) |U(j\omega)|^2 d\omega.$$

Hence, minimizing the output noise power for the worst case input noise is equivalent to designing a filter $U(s)$ with minimum \mathcal{H}_∞ norm. Therefore, in cases in which the input signal is subject to random disturbance with unknown but bounded power spectrum, the \mathcal{H}_∞ optimization approach should offer a more robust design.

2. SUMMARY OF DESIGN RESULTS

Let $\mathcal{L}_2(R_+)$ denote the Hilbert space of all real-valued Lebesgue measurable and square integrable functions $f(t)$ in R_+ . In [11] the time-domain Laguerre polynomials are defined as

$$l_n(t) = \frac{e^t}{n!} \frac{d^n}{dt^n} (e^{-t} t^n), \quad n = 0, 1, 2, \dots$$

For a given $p > 0$, the Laguerre functions with scale factor p are defined as

$$\phi_n^p(t) = \sqrt{2p} e^{-pt} l_n(2pt), \quad n = 0, 1, 2, \dots$$

It is known that the sequence $\{\phi_n^p(t)\}$ forms a uniformly bounded orthonormal basis for the Hilbert space $\mathcal{L}_2(R_+)$ (cf. [7], [11]).

Let \mathcal{H}_2 denote the Hardy space which consists of all complex-valued functions which are analytic and square integrable in the open right-half-plane with square integrable boundary functions. The \mathcal{H}_2 space Laguerre functions are defined as

$$\Phi_n^p(s) = \frac{\sqrt{2p}}{s+p} \left(\frac{s-p}{s+p} \right)^n, \quad n = 0, 1, 2, \dots \quad (1)$$

The sequence $\{\Phi_n^p(s)\}$ forms a uniformly bounded orthonormal basis for the Hilbert space \mathcal{H}_2 , see, [7].

Since Laguerre function sequence $\{\phi_n^p\}$ forms an orthonormal basis of $\mathcal{L}_2(R_+)$, any $u \in \mathcal{L}_2(R_+)$ can be expanded as

$$u(t) = \sum_{k=0}^{\infty} x_n \phi_n^p(t) \quad (2)$$

where $x_n = \int_0^\infty u(t) \phi_n(t) dt$ ($n = 0, 1, \dots$). Let U denote the Laplace transform of $u \in \mathcal{L}_2(R_+)$, then $U \in \mathcal{H}_2$. Since $\{\Phi_n^p\}$ forms an orthonormal basis of \mathcal{H}_2 , U can be expanded as

$$U(s) = \sum_{k=0}^{\infty} x_n \Phi_n^p(s) \quad (3)$$

Define

$$u_N(t) = \sum_{n=0}^{N-1} x_n \phi_n^p(t), \quad U_N(s) = \sum_{n=0}^{N-1} x_n \Phi_n^p(s) \quad (4)$$

Let us consider those filters whose Laplace transforms are stable real-rational transfer functions. The \mathcal{H}_∞ EC filtering problem, denoted as problem (P), can be posed as

$$\min \|U\|_\infty^2 \quad \text{subject to} \quad \varepsilon^-(t) \leq \psi_N(t) \leq \varepsilon^+(t), \quad \forall t \in R_+$$

where

$$\psi_N(t) = \int_0^\infty u_N(\tau) s(t-\tau) d\tau.$$

To avoid the trivial solution $U(s) = 0$, assume that there exists a $t_0 \in R_+$ such that $\varepsilon^-(t_0)\varepsilon^+(t_0) > 0$. Furthermore, let us assume that $\varepsilon^+(t) \geq \varepsilon^-(t)$, $t \in R_+$ since the problem has no solution otherwise.

By defining $x = [x_0, x_1, x_2, \dots, x_{N-1}]^T$, it can be verified that the output $\psi_N(t)$ can be written as

$$\psi_N(t) = \varphi^T(t) x$$

where $\varphi(t) = [y_0(t), y_1(t), y_2(t), \dots, y_{N-1}(t)]^T$ and $y_n(t) = \int_0^\infty \phi_n^p(\tau) s(t-\tau) d\tau$. It is clear that for a given input pulse $s(t)$, $\varphi(t)$ is a known vector-valued function. Define

$$\Omega_N \triangleq \{x \in R^N : \varepsilon^-(t) \leq \varphi^T(t) x \leq \varepsilon^+(t), \quad t \in R_+\} \quad (5)$$

Ω_N is a convex set which completely characterises the time-domain constraint defined in the EC filtering problem (P).

Through simple algebraic manipulation, the objective function $\|U\|_\infty^2$ can be written as

$$\|U\|_\infty^2 = \|R_a - X\|_\infty^2 \quad (6)$$

where

$$\begin{aligned} R_a(s) &= \sum_{n=0}^{N-1} x_{N-n-1} \frac{\sqrt{2p}}{s-p} \left(\frac{s+p}{s-p} \right)^n \\ X(s) &= - \sum_{n=0}^{\infty} x_{N+n} \frac{\sqrt{2p}}{s+p} \left(\frac{s-p}{s+p} \right)^n. \end{aligned}$$

Clearly, $X(s)$ is stable and $R_a(s)$ is antistable. For any given $x \in R^N$, $R_a(s)$ is a known real-rational transfer function. Let $W_c = W_c(x)$, $W_o = W_o(x)$ be, respectively, the controllability and observability matrices of $R_a(s)$ and $\lambda_i(W_c(x)W_o(x))$ the i th eigenvalue of $W_c(x)W_o(x)$. It is known that $\|U\|_\infty^2 = \max_i \lambda_i(W_c(x)W_o(x))$ (cf. [4]). Therefore, the minimization of the objective function defined in (P) can be expressed as

$$\min \|U\|_\infty^2 = \min_x \max_i \lambda_i(W_c(x)W_o(x))$$

This is a minimax problem.

To sum up, the EC filtering problem (P) can be recast as the following constrained \mathcal{H}_∞ model matching problem:

For any given integer $N > 0$, find a vector $x \in R^N$ and a stable transfer function $X(s)$ such that

$$\min_{x \in \Omega_N} \min_{X \in \mathcal{RH}_\infty} \|R_a - X\|_\infty^2 = \min_{x \in \Omega_N} \max_i \lambda_i(W_c(x)W_o(x))$$

This constrained minimax problem can be solved using the recently developed \mathcal{H}_∞ model matching theory and convex optimization techniques, see, e.g., [4], [10], [13].

Summarizing the above procedure in the form of an algorithm, we have

Algorithm

Step 1: Solve $\min_{x \in \Omega_N} \{\max_i \lambda_i(W_c(x)W_o(x))\}$ to obtain the minimizer $x^* = [x_0^*, x_1^*, x_2^*, \dots, x_{N-1}^*]^T$.

Step 2: Substitute x^* into R_a ; solve the unconstrained model matching problem $\min_X \|R_a - X\|_\infty$ to obtain X^* .

Step 3: Construct $U^*(s)$ by setting $U^*(s) = U_N(s) - \left(\frac{s-p}{s+p}\right)^N X^*(s)$.

Remark:

(i) Clearly, the major task to solve the constrained \mathcal{H}_∞ model-matching problem is to solve the equivalent minimax problem

$$\min_{x \in \Omega_N} \max_i \lambda_i(W_c(x)W_o(x))$$

Although this is a convex optimization problem defined on a convex set (see, e.g. [10]), it is in general nondifferentiable. Special numerical techniques are needed to perform the optimization task for large N . In [9] and [10] it is suggested that the ellipsoid algorithm combined with vector Lanczos procedure can be used to solve the minimax problem efficiently. Alternatively, it can also be solved using semidefinite programming techniques [13]. It is worth noting that in many situations minimizing the Euclidean norm

of x or the trace of $W_c(x)W_o(x)$ (which is much easier to do) often yields a satisfactory suboptimal solution.

(ii) Note that the time-domain constraints are imposed on ψ_N instead of ψ . This means that, due to the addition term $X(s)$, the output response $\psi(t)$ of the filter $U(s)$ to the given input signal $s(t)$ could violate the envelope constraints. However, by using Theorem 2.1 the constraint violations can be made as small as desired by choosing N sufficiently large. Also, if small increase of the filter's \mathcal{H}_∞ norm is permitted, the constraint violations can be controlled by using pole placement techniques (see, [9], [10]).

(iii) From \mathcal{H}_∞ model-matching theory we know (see [4]) that in general the order of the optimal filter $U^*(s)$ is $2N$. Using existing model reduction techniques such as balanced realization truncation or optimal Hankel-norm approximations, a lower order suboptimal filter can be obtained for a prespecified level of small output constraint violation. This will be illustrated in the simulation section.

Let us introduce the following notation: $g_n = O(f_n)$ means that there exists a constant $C > 0$ and an integer $N > 0$ such that $|g_n| \leq C|f_n|$ for any $n \geq N$. The following \mathcal{H}_∞ approximation result can be established:

Theorem 2.1 Given an input signal $s(t) \in \mathcal{L}_2(R_+)$, let $U^*(s)$, with impulse response $u^*(t)$, denote the solution of the following EC filtering problem

$$\min \|U\|_\infty^2 \text{ subject to } \varepsilon^-(t) \leq \psi(t) \leq \varepsilon^+(t), \quad \forall t \in R_+ \quad (7)$$

where

$$\psi(t) = \int_0^\infty s(\tau)u(t-\tau)d\tau.$$

If U^* is a stable real-rational function, then there exists a positive constant $\rho < 1$ such that

$$\|U^* - U_N^*\|_\infty = O(\rho^N)$$

Furthermore, for any $\delta_1 > 0$, $\delta_2 > 0$, there exists an $N_0 > 0$ such that for all $N \geq N_0$,

$$\|U^*\|_\infty - \delta_1 \leq \|U_N^*\|_\infty \leq \|U^*\|_\infty + \delta_1$$

$$\varepsilon^-(t) - \delta_2 \leq \psi_N^*(t) \leq \varepsilon^+(t) + \delta_2$$

Remark: Theorem 2.1 shows that U_N converges to U exponentially fast. It also shows that the first N th partial sum U_N can provide a good suboptimal solution to the \mathcal{H}_∞ optimal EC filtering problem (7).

3. NUMERICAL EXAMPLE

Let us now apply the method presented in previous section to the design of an equalization filter for a digital transmission channel consisting of a coaxial cable on which data is transmitted according to the DX3 standard (see [2]). For this filter design problem, both the input signal (the impulse response of a coaxial cable) and the output envelope are given in continuous time domain. For computational purpose, we shall discretize both the input signal and the output mask (note that the approach presented in Section

2 does not require the discretization of either the input signal or the output mask). The design objective is to find an equalizing filter which takes a sampled impulse response of a coaxial cable with a loss of 30dB at a normalized frequency of $1/T$ as input and produces an output which lies within the envelope given by the DSX-3 pulse template (cf. [2]). To have a good representation of the input signal, the sampling rate should be fast enough. In our numerical studies, 1024 samples are used over the normalized time interval of $[0, 32T]$, i.e. the sampling period is $T/32$. The simulation results are now summarized as follows: Applying the Algorithm of Section 2 to this filter design problem, we obtain a 12th order LTI filter. Through model reduction (using balanced truncation method), a 5th order LTI model is obtained. Fig. 2 is the plot of the sampled input signal and output response. The plot of magnitude spectra of the input-output signals is shown in Fig. 3.

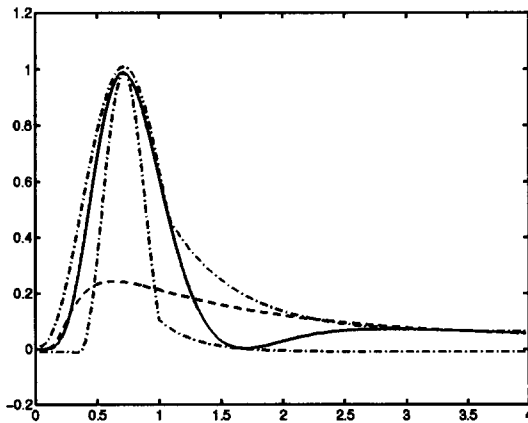


Figure 2. Plot of the sampled time domain output signal (solid line), input signal (dashed line), and the output mask (dash-dotted lines).

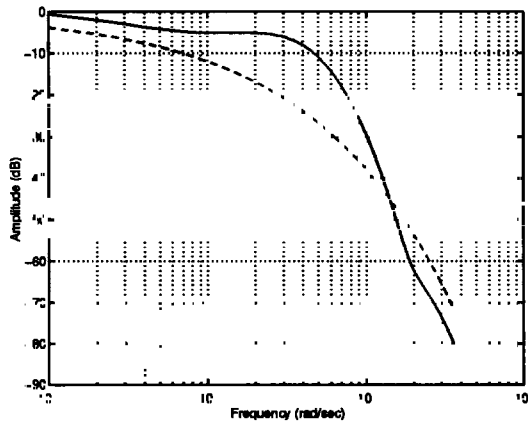


Figure 3. Magnitude spectrum plot of the output signal (solid line) and the corresponding input signal (dashed line).

Remark: It is evident from Fig. 2 that the time domain output signal fits into the output mask. From a different perspective it is evident from Fig. 3 that equalization in terms of designing a transfer function with a flatter frequency response has also been achieved. However, it can be easily seen from Fig. 2 that part of the output signal at

certain points actually meets the output mask. This means that any disturbance at the input of the channel or any filter implementation errors could cause the output to violate the envelope constraint. The modification of the proposed algorithms to achieve a robust design with respect to time domain constraints is a current topic of research.

Extensive simulations indicate that for the worst case input noise (cf. [4]) the suboptimal H_∞ EC filter can achieve between 30 to 40 percent better performance than the same order optimal L_2 EC filter in terms of output noise power reduction.

REFERENCES

- [1] K.M. Ahmed and R.J. Evans, "An adaptive array processor with robustness and broadband capabilities," *IEEE Trans. Antennas and Propagat.*, vol. AP-32, pp. 944-950, 1984.
- [2] Bell Communications, "DSX-3 isolated pulse template and equations," *Technical Reference TR-TSY-000499*, Issue 2, December, pp. 9-17, 1988.
- [3] R.J. Evans, A. Cantoni, and T.E. Fortman, "Envelope-constrained filter, Part I, Theory and application; Part II, adaptive structures," *IEEE Trans. Information Theory*, vol. IT-23, pp. 421-444, 1977.
- [4] B.A. Francis, *A Course in H_∞ Control Theory*, Springer-Verlag, Berlin, Heidelberg, 1987.
- [5] Y.W. Lee, *Statistical Theory of Communication*. New York: Wiley, 1960.
- [6] R. Lucky, J. Salz, and E.J. Weldon, *Principles of data communications*, New York: McGraw-Hill, 1968.
- [7] P.M. Mäkilä, "Laguerre series approximation of infinite dimensional systems," *Automatica*, vol. 26, pp. 985-995, 1990.
- [8] R.J. McAulay and J.R. Johnson, "Optimal mismatched filter design for radar ranging detection and resolution," *IEEE Trans. Information Theory*, vol. IT-17, pp. 696-701, 1971.
- [9] H. Rotstein and A. Sideris, "Optimal mismatched networks for a class of discrete-time systems," *IEEE Trans. Signal Processing*, vol. 39, pp. 2104-2108, 1991.
- [10] H. Rotstein and A. Sideris, " H_∞ optimization with time-domain constraints," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 762-779, 1994.
- [11] A. Sideris and H. Rotstein, "Single-input-single-output H_∞ -control with time domain constraints," *Automatica*, Vol. 29, No. 4, pp. 969-983.
- [12] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Coll. Publ., vol. 23, 1939.
- [13] K.L. Teo, A. Cantoni, and X.G. Lin, "A new approach to optimization of envelope constrained filters with uncertain input," *IEEE Trans. Signal Processing*, vol. 42, pp. 426-429, 1994.
- [14] L. Vandenberghe and S. Boyd, "Semidefinite programming," *SIAM Review*, March, 1996.
- [15] W. Zheng, A. Cantoni, B. Vo and K.L. Teo, "Recursive procedures for constrained optimization problems and its application in signal processing", *IEE Proc.-Vision, Image and Signal Processing*, vol. 142, pp. 161-168, 1995.