COMPLETE ITERATIVE RECONSTRUCTION ALGORITHMS FOR IRREGULARLY SAMPLED DATA IN SPLINE-LIKE SPACES

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ABSTRACT

We prove that the exact reconstruction of a function s from its samples $s(x_i)$ on any "sufficiently dense" sampling set $\{x_i\}_{i\in I} \subset \mathcal{R}^n$, where I is a countable indexing set, can be obtained for a large class of spline-like spaces that belong to $L^p(\mathcal{R}^n)$. Moreover, The reconstruction can be implemented using fast algorithms. Since, a special case is the space of bandlimited functions, our result generalizes the classical Shannon-whittacker sampling theorem on regular sampling and the Paley-Wiener theorem on nonuniform sampling.

1. INTRODUCTION

In sampling theory, the main goal is the exact reconstruction of a continuous function $g(x) \in C(\mathcal{R}^n)$ form its samples $g(x_i)$ on a sampling set $X = \{x_i\}_{i\in I} \subset \mathcal{R}^n$. If the sampling is uniform, i.e., the set $X = \{x_i\}_{i\in I}$ lies on a uniform cartesian grid, then the function $g(x)$ can be recovered exactly from its samples as long as $g(x) \in L^2(\mathcal{R}^n)$ is bandlimited and that the grid-points density is larger than the Nyquist density [21]. This is the classical Shannon-Whittacker sampling theorem. In particular, if $g \in L^2(\mathcal{R})$ and its Fourier transform $\hat{g}(f) = \int g(x)e^{-i2\pi fx}dx$ is such that $\hat{g}(f) = 0, \forall f \notin I = [-\frac{1}{2}, \frac{1}{2}]$ (i.e., $g \in B_{\frac{1}{2}}$), then $g(x)$ can be recovered from $g(x_0 + k)$, $k \in \mathcal{Z}$ by the formula [21, 24]:

$$g(x) = \sum_{k\in\mathcal{Z}} g(x_0 + k)\text{sinc}(x - x_0 - k), \qquad (1)$$

where $x_0 \in \mathcal{R}$ is arbitrary, and where, the interpolating function $\text{sinc}(x) = \frac{\sin(\pi x)}{x}$ is simply the inverse Fourier transform of the ideal filter function $\chi_I(f)$ in the band $I = [-\frac{1}{2}, \frac{1}{2}]$ (i.e., $\chi_I(f)$ is the characteristic function of the interval I: $\chi_I(f) = 1$, $\forall f \in I$, and $\chi_I(f) = 0$, $\forall f \notin I$). An identical statement to the Shannon-Whittacker sampling theorem is that the set of bandlimited functions $B_{\frac{1}{2}}$, with bandwidth in $[-\frac{1}{2}, \frac{1}{2}]$, is precisely the set of functions belonging to the space $S(\text{sinc})$ that is obtained by linear combinations of the sinc-function and its integer shifts, with square summable coefficients:

$$S(\text{sinc}) = \left\{ \sum_{k\in\mathcal{Z}} c(k)\text{sinc}(x - k) \mid c \in \ell^2 \right\}. \qquad (2)$$

In fact, the set $\{\text{sinc}(x - k)\}_{k\in\mathcal{Z}}$ is an (even orthonormal) unconditional basis for the space $B_{\frac{1}{2}}$. Specifically, a basis $\{e_k\}_{k\in\mathcal{Z}}$ of a Banach space $\mathcal{B}$ (a Banach space is an infinite dimensional normed vector space which is complete) is unconditional if $\sum_k c(k)e_k \in \mathcal{B}$ implies that $\sum_k \epsilon_k c(k)e_k \in \mathcal{B}$ for any choice of ϵ_k equal to either $+1$ or -1 [17, pp. 16].

The Shannon-Whittacker sampling theorem can be generalized by changing the sinc-function in (2) to an appropriate generating function $\lambda(x)$ as described in [3, 4, 5]:

$$S(\lambda) = \left\{ \sum_{k\in\mathcal{Z}} c(k)\lambda(x - k) \mid c \in \ell^2 \right\}. \qquad (3)$$

In this generalization, the spaces $S(\lambda)$ are not necessarily bandlimited. The bandlimited case and the expansion (2) are obtained as the special case $\lambda(x) = \text{sinc}(x)$. Moreover, the sampling theory for bandlimited functions is also a limit case for families $\lambda^n(x)$ with increasing smoothness as $n \to \infty$ (see [5] for details, and [6, 23] for examples that use polynomial spline functions). Other generalizations of the uniform sampling theory are discussed in [15, 16].

The classical result for nonuniform sampling is due to Paley and Wiener, and it states that if the set $X = \{x_i\}_{i\in\mathcal{Z}} \subset \mathcal{R}$ is such that $|x_i - i| < \pi^{-2}$, then a bandlimited function $g \in C(\mathcal{R})$, with bandwidth $[-\gamma, \gamma]$ and $|\gamma| < \frac{1}{2}$, can be completely recovered from its samples $g(x_i)$ [19]. Kadec later showed that the same is true if $|x_i - i| < 1/4$, cf. [26]. A detailed exposition and other generalizations of the Paley-Wiener irregular sampling result can be found in [7]. There are also other types of results on irregular sampling in which the sampling set is nonuniform, but fixed [25], i.e., the reconstruction is guaranteed for a specific sampling set only, and no others. However, these types of results are different from those of Paley-Wiener which do not require a fixed sampling set, but any sufficiently dense sampling set.

Extension to the multidimensional irregular sampling of bandlimited functions in L^p-spaces can be found in [14, 11] (a function g is in L^p if $\int |g(x)|^p dx < \infty$). The results are based on the properties of Wiener amalgam spaces [10, 11] which we also use in this manuscript. In particular, we will use the Wiener spaces $W^p = W(C, L^p)$ which are locally continuous and globally L^p. This means that a function g belongs to W^p if g is continuous, bounded, and $\int |g(x)|^p dx < \infty$.

In the present paper, we will extend the theory of multidimensional irregular sampling in [14, 11] to the case of spline-like spaces $S(\lambda) \subset L^p(\mathcal{R}^n)$ of the form

$$S(\lambda) = \left\{ \sum_{k \in \mathcal{Z}^n} c(k)\lambda(x-k) \mid c \in \ell^p(\mathcal{R}^n) \right\}. \quad (4)$$

Our results can also be viewed as an extensions of those in [18] where some ideas from [14, 11], and ideas similar to those in [5] are used to construct, under restrictive conditions, an irregular sampling theory for polynomial spline and other wavelet subspaces of $L_2(\mathcal{R})$.

2. SPLINE-LIKE SPACES

Since the space $S(\lambda)$ that we consider must belong to $L^p(\mathcal{R}^n)$, the function λ cannot be chosen arbitrarily. Moreover, we want the set $\{\lambda(x-k)\}_{k\in\mathcal{Z}^n}$ to form an unconditional basis of $S(\lambda)$. A sufficient condition is given by the following proposition:

Proposition 2.1 *The space $S(\lambda)$ is closed, and the set $\{\lambda(x-k)\}_{k\in\mathcal{Z}^n}$ forms an unconditional basis of $S(\lambda)$ if there exist two constants $B \geq A > 0$ such that*

$$A \|c\|_{\ell^p}^p \leq \left\| \sum_{k\in\mathcal{Z}^n} c(k)\lambda(x-k) \right\|_{L^p}^p \leq B \|c\|_{\ell^p}^p. \quad (5)$$

To see that $\{\lambda(x-k)\}_{k\in\mathcal{Z}^n}$ is a basis, simply note that if $\sum_k c(k)\lambda(x-k) = 0$, then the left inequality of (5) implies that $c = 0$. The inequality on the rigth of (5) implies that this basis is unconditional. For the special case of $L^2(\mathcal{R}^n)$, to satisfy (5), a necessary and sufficient condition on a function λ is that the Fourier transform

$$\begin{aligned} \hat{a}(f) &= \sum_{k\in\mathcal{Z}^n} a(k)e^{-i2\pi fk} \quad (6) \\ &= \sum_k \left|\hat{\lambda}(f+k)\right|^2 \end{aligned}$$

of the sampled autocorrelation $a(k) = (\lambda * \lambda^\vee)(k)$, $k \in \mathcal{Z}^n$, must be uniformly bounded above and below (here by definition $\lambda^\vee(x) = \lambda(-x)$), i.e., there exist two constants $M \geq m > 0$ such that [4, 5]

$$m \leq \hat{a}(f) = \sum_{k\in\mathcal{Z}^n} \left|\hat{\lambda}(f+k)\right|^2 \leq M \quad a.e. \quad (7)$$

This result can be found in [4, 5], and is also a special case of a general result in [2].

3. MAIN RESULT ON EXACT RECONSTRUCTION

Our ability to reconstruct a function $g \in S(\lambda)$ from irregularly spaced samples depends on the sampling density of the sampling set. In particular, for any sampling set $X = \{x_i\}_{i\in I} \subset \mathcal{R}^n$, the density measure that we use is given by [12, 13]:

Definition 3.1 *A set $X = \{x_i\}_{i\in I}$ is γ-dense in $\mathcal{R}^n$, if $\mathcal{R}^n$ is the union of balls centered on x_i, and of radius γ:*

$$\mathcal{R}^n = \bigcup_{i\in I} B_\gamma(x_i).$$

In one dimension, a set $X = \{x_i\}_{i\in I} \subset \mathcal{R}$ is γ-dense if the maximal distance $|x_{i+1} - x_i|$ between any two consecutive sampling points is smaller that γ. The γ-density plays the same role for irregular sampling as the role of the Nyquist rate for uniform sampling.

If the set $X = \{x_i\}_{i\in I}$ is γ-dense, then we can always construct a piecewise-constant function $V_X s$ that interpolates the samples $s(x_i)$ of a function $s \in S$, i.e., $V_X s(x_i) = s(x_i)$, $\forall i \in I$. The interpolating function that we construct is constant on the Vornoi domains V_i of the sampling points x_i. These domains are defined by

$$V_i := \{x : |x_i - x| < |x_j - x| \ \forall i \neq j\}.$$

In particular, in one dimension, the Vornoi domain of the point x_i is the interval $[m_i, m_{i+1}]$, where m_i is the midpoint between x_i and x_{i-1}: $m_i = (x_i + x_{i-1})/2$. Thus, for a function $s \in W^p$, we define the piecewise-constant interpolant operator V_X by

$$V_X s = \sum_{i\in I} s(x_i)\chi_{V_i} \quad (8)$$

where χ_{V_i} is the characteristic function of V_i. It is well-known that the operator V_X maps the space W^p into the space $W(L^\infty, L^p) \subset L^p$ [9].

By interpolating the samples of a function $s \in S$ with V_X, and then projecting the interpolated function $V_X s$ on the space S, we get an approximation $s_1 \in S$ of our original function s. Since s and s_1 belong to S, the error $e = s - s_1$ belongs to S as well. The values

$e(x_i) = s(x_i) - s_1(x_i)$ of the error at the sampling points can be evaluated. Using the samples $e(x_i)$, we can repeat the interpolation and projection procedure to obtain a function $e_1 \in S$. We add e_1 to s_1 to obtain the new approximation s_2 to our function s. By repeating this procedure, we obtain a sequence $s_1 + e_1 + e_2 + e_3 + \cdots$ that converges to the function s as stated in the theorem below (for proof, cf. [1]). The operator P is simply a linear projector (not necessary orthogonal) from L^p into S.

Theorem 3.1 *If the generating function λ belongs to the Wiener space W^1, then we can recover $s \in S(\lambda)$ from its samples $\{s(x_i)\}$ on any γ-dense set $X = \{x_i\}$, for a sufficiently small γ, by the following iterative algorithm*

$$s_{n+1} = PV_X(s - s_n) + s_n \quad (9)$$
$$s_0 = PV_X s \quad (10)$$

where P is any projector, and where the convergence of s_n to s occurs in the L^p-norm and the W^p-norm, and we have $\|s - s_n\| \leq C_1 \alpha^n$, for some $\alpha = \alpha(\gamma) < 1$, and $C_1 < \infty$.

A typical example for the performance of the suggested allgorithm is given in Figure 1. Although most examples are "approximately band-limited" the use of standard band-limited reconstruction methods would give only approximate reconstruction, whereas we have *complete* reconstruction under the circumstances described in this note. In effect, the iterative algorithm (9) reduces to

$$s_{n+1} = (I + T + T^2 + \cdots + T^n)PV_X s$$

where $T = I - PV_X$, and gives the inversion of $I - T$ by the Neumann series: $(I - T)^{-1} = I + T + T^2 + \cdots$

Remark 3.1 *The fact that the sampling values $s(x_i)$ are well defined follows from the fact that, under our condition on the generating function λ, the space $S(\lambda)$ is a subspace of the locally continuous and globally L^p function space W^p.*

Remark 3.2 *An important point is that the reconstruction does not depend on any individual sampling set X, but only on the their γ-density. This means that as long as the gap between samples is not too large, we can always recover any function in S exactly with our iterative algorithm. Moreover, in contrast to other appraoches, our method can handle clusters in the sampling set. On the other hand it is not a simple frame algorithm, since we are not in a Hilbert space setting anymore, and thus the alternative tools, such as Wiener amalgam spaces, have to be used.*

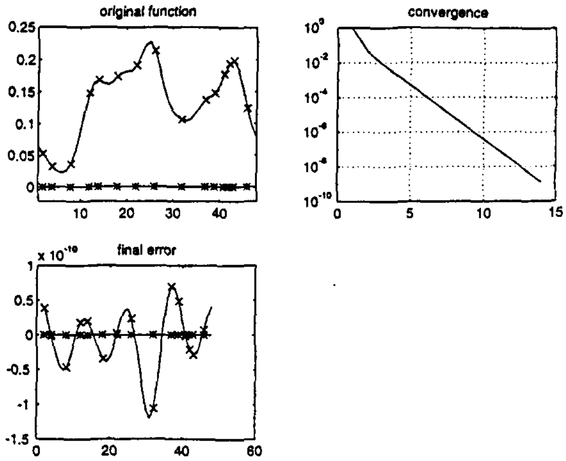

Figure 1: The function $s \in S(Gaussian)$ (top left) which is a linear combination of a Gaussian function and its integer shifts, is sampled at the points marked by a cross $\times$ (clearly, it is not a bandlimited function). The bottom left panel shows the error after 15 iterations. The bottom right panel clearly shows that the error decreases exponentially fast.

Remark 3.3 *The interpolant V_X can be replaced by other types of (quasi-)interpolants without changing the validity of the theorem. In particular, we can choose a set of measurable functions $\Phi = \{\varphi_i\}_{i \in I}$ associated with the sampling set X and satifying the following three properties: (1) $0 \leq \varphi_i \leq 1$, $\forall i \in I$, (2) support $\varphi_i \subset B_\gamma(x_i)$, (3) $\sum_{i \in I} \varphi_i = 1$ and use an interpolation of the form $Q_\Phi s = \sum_{i \in I} s(x_i)\varphi_i$.*

4. CONCLUSION

We have shown how to reconstruct spline-like functions exactly, from their irregular samples. These functions are not bandlimited, in general. The bandlimited theory is a special case. The theory is valid for any dimension and it generalizes the Paley-Wiener theory on nonuniform sampling. Theorem 3.1 shows that, the reconstruction algorithm converges exponentially fast ($O(\alpha^n)$, $\alpha < 1$) as the number of iterations n increases. Since the contraction factor α is a decreasing function of the density γ the algorithm will converge more rapidly for denser sets. Moreover, for the special case of finite energy signals L^2, the projection operator in the algorithm can always be implemented with fast filtering algorithms as described in [5], which further improves the speed of the algorithm.

5. REFERENCES

[1] A. Aldroubi and H. G. Feichtinger. Exact iterative reconstruction algorithm for multivariate irregularly sampled functions in spline-like spaces: The L^p-Theory. preprint.

[2] A. Aldroubi. Oblique projections in atomic spaces. *Proc. Amer. Math. Soc.*, 124:2051–2060, 1996.

[3] A. Aldroubi and M. Unser. Families of wavelet transforms in connection with Shannon's sampling theory and the Gabor transform. In [8], pages 509–528. Academic Press, 1992.

[4] A. Aldroubi and M. Unser. Families of multiresolution and wavelet spaces with optimal properties. *Numer. Funct. Anal. and Optimiz.*, 14(5):417–446, 1993.

[5] A. Aldroubi and M. Unser. Sampling procedure in function spaces and asymptotic equivalence with Shannon's sampling theory. *Numer. Funct. Anal. and Optimiz.*, 15(1):1–21, 1994.

[6] A. Aldroubi, M. Unser, and M. Eden. Cardinal spline filters: Stability and convergence to the ideal sinc interpolator. *Signal Processing*, 28:127–138, 1992.

[7] J.J. Benedetto. Irregular sampling and frames. In [8], pages 445–507. 1992.

[8] C.K. Chui, editor. *Wavelets: A Tutorial in Theory and Applications.* Academic Press, San Diego, CA, 1992.

[9] H.G. Feichtinger. Banach convolution algebras of wiener type. In *Proc. Conf. Functions, Series, Operators, Budapest*, pages 509–524, Amsterdam-Oxford-New York, 1980. Colloquia Math. Soc. J. Bolyai, North Holland Publ. Co.

[10] H.G. Feichtinger. Generalized amalgams, with applications to Fourier transform. *Can. J. of Math.*, 42(3):395–409, 1990.

[11] H.G. Feichtinger. New results on regular and irregular sampling based on Wiener amalgams. In K. Jarosz, editor, *Proc. Conf. Function Spaces*, volume 136 of *Lect. Notes in Math*, pages 107–122, Edwardsville, IL, April 1990, 1991. Marcel Dekker.

[12] H.G. Feichtinger. Wiener amalgams over Euclidean spaces and some of their applications. In K. Jarosz, editor, *Proc. Conf. Function Spaces*, volume 136 of *Lect. Notes in Math.*, pages 123–137, Edwardsville, IL, April 1990, 1991. Marcel Dekker.

[13] H.G. Feichtinger and K. Gröchenig. Non-orthogonal wavelet and Gabor expansions, and group representations. In [20], pages 353–375. 19.

[14] H.G. Feichtinger and K. Gröchenig. Iterative reconstruction of multivariate band-limited functions from irregular sampling values. *SIAM J. Math. Anal. 231*, pages 244–261, 1992.

[15] J.R. Higgins. Five short stories about the cardinal series. *Bull. Amer. Math. Soc.*, 121:45–89, 1985.

[16] A.J. Jerri. The Shannon sampling theorem-its various extensions and applications: A tutorial review. Proc. IEEE, 65:1565–1596, 1977.

[17] J. Lindenstrauss and L. Tzafiri. *Cassical Banach Spaces.* Springer-Verlag, Berlin, 1973.

[18] Y. Liu. Irregular sampling for spline wavelet subspaces. *IEEE Trans. Inform. Theory*, 42(2):623–627, 1996.

[19] R.E.A.C Paley and N. Wiener. Fourier transform in the complex domain. In *Amer. Math. Soc. Colloquium publications.* Amer. Math. Soc., 1934.

[20] M.B. Ruskai, G. Beylkin, R.R. Coifman, I. Daubechies, S. Mallat, Y. Meyer, and L. Raphael, editors. *Wavelets and their Applications.* Jones and Bartlett, Boston, 1992.

[21] C.E. Shannon. Communications in the presence of noise. *Proc. IRE*, 37:10–21, 1949.

[22] M. Unser and A. Aldroubi. A general sampling theory for non-ideal acquisition devices. *IEEE Trans. on Signal Processing*, 42(11):2915–2925, 1994.

[23] M. Unser, A. Aldroubi, and M. Eden. Polynomial spline signal approximations: filter design and asymptotic equivalence with Shannon's sampling theorem. *IEEE Trans. Image Process.*, 38:95–103, 1991.

[24] J.M. Whittacker. *Interpolation Function Theory.*

[25] K. Yao. Application of Reproducing Kernel Hilbert Spaces-bandlimited signal models. Information and Control, 11:429–444, 1967.

[26] R.K. Young. *Wavelet Theory and its Applications.* Number SECS 189 in Kluwer international series in engineering and computer science. Kluwer Academic Publishers, 1992.

