SIGNAL RECONSTRUCTION FROM PHASE ONLY INFORMATION AND APPLICATION TO BLIND SYSTEM ESTIMATION

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ABSTRACT

We propose a method for the reconstruction of a complex signal from its Fourier phase only, where the phase is known within a linear phase term, and the sequence's length is unknown. The case of the phase known exactly has received a lot of attention in the past, however, in most cases the phase can be estimated up to a linear phase term whose slope is unknown. Moreover, in most cases of interest, the exact length of the sequence which is to be recovered is unknown. As an application of the reconstruction from phase technique, we propose a method for blind channel identification.

1. INTRODUCTION

Reconstruction from phase only has received a lot of attention due to its applicability to important practical problems. In many cases the desired signal has been distorted by some mechanism that is known to affect the signal's Fourier magnitude only. Such situations occur approximately in long-term exposure to atmospheric turbulence or when images are blurred by defocused lenses with circular aperture stops [1]. As a consequence, the observed spectrum is distorted only magnitude-wise.

It has been established [2] that a real FIR sequence that has no zeros on the unit circle and in conjugate reciprocal pairs can be reconstructed within a scale factor from its Fourier phase only, and several reconstruction approaches have been proposed [2], ([4] and references therein). These methods assume that the true phase, or its principal argument, is available, and that the length and the region of support of the sequence to be recovered is known. Most of them are iterative schemes that during each iteration force an initial estimate of the sequence to be nonzero only within its known region of

support, and have the given phase. Most phase estimation procedures, however, return the phase within a linear phase term, which is unknown and thus shifts by an unknown amount the region of support of the sequence. The case of unknown length has been addressed in the past through rank determination [2], which is a sensitive task when slight disturbances are present in the data.

In this paper we show that a complex FIR sequence which does not contain zero-phase convolutional components and does not have zeros on the unit circle is uniquely characterized up to a complex scalar by its Fourier-phase only. A reconstruction procedure is given in Section 2. In Section 3 the proposed method is applied for blind system identification. Simulation results are given in Section 4.

2. SIGNAL RECONSTRUCTION FROM PHASE

Proposition 1 *A FIR sequence which does not contain zero-phase convolutional components and has no zeros on the unit circle is uniquely characterized up to a scalar by its Fourier-phase only.*

Proof :
The proof for a real FIR sequence can be found in [2]. For the case of a complex sequence the proof can be modified as follows. Let $x(k)$ and $y(k)$ be two FIR sequences with no zeros on the unit circle, which have the same Fourier phases, and do not contain zero-phase convolutional components, and let $X(z)$, $Y(z)$ be the corresponding Z-transforms. The sequence corresponding to

$$G(z) = X(z)Y^*(1/z^*) \qquad (1)$$

is a zero-phase sequence. Let z_0 be a zero of $X(z)$. Then z_0 is also a zero of $G(z)$ and since $G(z)$ is zero-phase, $1/z_0^*$ is a zero of $G(z)$ too. Since the zeros of

$G(z)$ are the zeros of $X(z)$ and $Y^*(1/z^*)$, $1/z_0^*$ must be a zero of either $X(z)$ or $Y^*(1/z^*)$. The sequence $x(k)$ does not contain zero-phase convolutional components, thus $1/z_0^*$ cannot be its zero, therefore $1/z_0^*$ is a zero of $Y^*(1/z^*)$. Equivalently, z_0 is a zero of $Y(z)$, i.e., the FIR sequences $X(z)$ and $Y(z)$ have the same zeros, or equivalently, $x(k) = cy(k)$, where c is a constant. □

A procedure for system reconstruction from discretized phase is outlined in the following. Let $x(k), k = 0, ..., N - 1$ be a FIR, generally complex sequence that has no zeros on the unit circle, does not contain zero-phase convolutional components, and let $\theta_x(\omega)$ be its Fourier-phase. Initially we will assume that the length N is known, and later we consider the case of unknown length.

From the fact that the phase of $X(\omega)$ equals $\theta_x(\omega)$ we get:

$$\frac{Im[X(\omega)]}{Re[X(\omega)]} = tan(\theta_x(\omega)). \quad (2)$$

Expressing $X(\omega)$ in terms of $x(k)$ and after some mathematical manipulations we get:

$$\sum_{k=1}^{N-1} x^R(k)sin(\omega n - \theta_x(\omega))$$

$$-\sum_{k=0}^{N-1} x^I(k)sin(\omega n + \theta_x'(\omega)) = sin(\theta_x(\omega)), \quad (3)$$

where $x^I(k)$, $x^R(k)$ are the imaginary and real parts of $x(k)$,

$$\theta_x'(\omega) = sgn\{\theta_x(\omega)\}\frac{\pi}{2} - \theta_x(\omega), \quad (4)$$

and was taken $x^R(0) = 1$. Equation (3) is an extension of a similar expression presented in [2] for the case of a real signal.

Evaluating (3) at L discrete frequencies $\{\omega = \frac{2\pi}{L}k, \ k = 0, .., L - 1\}$, we can form the system of equations:

$$\Theta_x \hat{x} = \theta_x, \quad (5)$$

where

$$\hat{x} = [x^R(1), ..., x^R(N-1), x^I(0), x^I(1), ..., x^I(N-1)] \quad (6)$$

$$\theta_x = [sin(\theta_x(0)), sin(\theta_x(\frac{2\pi}{L})), ..., sin(\theta_x(\frac{2\pi}{L}(L-1)))]^T, \quad (7)$$

and Θ_x can be easily inferred from (3).

Since there are $2N - 1$ unknowns to be estimated, L must be chosen so that $L > 2N-1$. Actually any set of discrete frequencies can be used to form the matrix Θ_x as long as the number of these frequencies is greater that $2N - 1$.

The matrix Θ_x has full (column) rank [2] and the least squares solution of (5) is

$$\hat{x}_x = (\Theta_x^T \Theta_x)^{-1} \Theta_x^T \theta_x. \quad (8)$$

An adaptive solution can also be obtained via LMS algorithm. It should be noted that it is only the principal argument of the phase that is required in (5), since the phase appears inside a sine term.

Unknown linear phase term

If $\theta_x(\omega)$ is known within a linear phase term $k\omega$, where k is an unknown integer in the range $[N_1, ..., N_2]$, prior to applying (8) a linear term $l\omega$ must be subtracted from $\theta_x(\omega)$, where l is an integer in the range $[N_1, ..., N_2]$. If a solution is found it will be equal to $x(k)$ within a scalar constant. Let $\hat{x}_l$ the least squares (LS) solution of (5) corresponding to phase $\theta_x(\omega) - l\omega$. To check if $\hat{x}_l$ is an acceptable solution we can check how small the LS error is, or as a more robust way, we propose to check if the difference between its phase and $\theta_x(\omega)$ is a straight line in ω.

Unknown length N

It can be shown that the rank of a matrix similar to Θ_x of (5) equals $2N-1$, where N is the sequence length. Thus we can always overestimate the length, form the matrix Θ_x and determine its rank. Rank determination, however, is not an easy task in the presence of even slight noise in the data.

Two FIR sequences corresponding to the same Fourier-phase and different lengths differ by a zero-phase convolutional component. If we apply the procedure outlined above to estimate $x(k)$ from $\theta_x(\omega)$ based on a length less that N we will find no solution [5]. For length greater that N we will find a solution that is equal to the convolution of $x(k)$ with a zero-phase sequence. The length of this zero-phase sequence will be equal to the amount of the length overestimate. Thus, starting from a small length and keep increasing it, the first solution found is the right one.

Unknown linear phase term and unknown length

The procedure for the reconstruction of $x(k)$ from its phase known within a linear phase term ωk is summarized as follows. Let N be the true length of $x(k)$, l be a guess for it, and $\theta_x(\omega)$ be the estimated phase. The following loop is implemented:
For $l = 2, 3, ...$
    For $m = -(l - 1) : l - 1$
        Evaluate (8) based on the phase

$$\theta_m(\omega) = \theta_x(\omega) - m\omega, \quad (9)$$

and let $\mathbf{x}_{m,l}$ be the corresponding solution. Let the Fourier-phase of the solution be $\theta_{x_{m,l}}(\omega)$. If $e(\omega) \triangleq \theta_x(\omega) - \theta_{x_{m,l}}(\omega)$ is a straight line in ω then $\mathbf{x}_{m,l}$ is the right solution, i.e., $x_{m,l}(k) = x(k)$, and the iteration stops here. If not, then the loop in m proceeds for the next value of m.

        end

end

To determine whether $e(\omega)$ is a straight line, we propose to perform a least-squares fit of $e(\omega)$ to the equation of a line. Let $E(l, m)$ be the LS error of the fit corresponding to length l and phase $\theta_m(\omega)$, and let $E(l) = \min_m E(l, m)$.

For each value of l, the reconstructed sequence corresponding to the location of the minimum in $E(l, m)$ is a potential solution. For $l \geq N$ the LS error exhibits a profound minimum that stays almost constant as l increases, and which is several orders or magnitude smaller that the minima corresponding to $l < N$. We can monitor $E(l)$ as l increases, and when it stabilizes to a low value, i.e., $E(l_0) \approx E(l_0 + 1) \approx E(l_0 + 2)...$ choose as solution the one corresponding to length l_0.

3. APPLICATION TO BLIND CHANNEL ESTIMATION

Consider the single input two output (or two-channel) problem

$$x_i(k) = s(k) * h_i(k) + n_i(k), \ i = 1, 2, \qquad (10)$$

where

$$s(k) = e(k) * r(k) \qquad (11)$$

is a zero-mean, stationary, generally non-white process, $e(k)$ is a white random process and $r(k)$ is a deterministic sequence; $h_i(k)$, $i = 1, 2$ correspond to the channel responses and are assumed to have no common zeros; and $n_i(k)$, $i = 1, 2$ are noise processes uncorrelated to each other and to $s(k)$. It is also assumed that there are no common zeros and no zero-pole cancellations between $h_i(k)$ and convolutional components of $s(k)$.

Let $g_i(k) = r(k) * h_i(k)$ and let $\tilde{g}_i(k)$ denote the minimum-phase equivalent of $x_i(k)$ and $\tilde{G}_i(\omega)$ its Fourier Transform $(i = 1, 2)$. In the noise free case the minimum-phase equivalent can be obtained by applying any autocorrelation based system identification method on $x_i(k)$, [3]. In the noisy case, a similar procedure will return an estimate of the minimum-phase equivalent. Let [6]

$$
\begin{aligned}
S_{min}(\omega) &= S_{x_1 x_2}(\omega)\tilde{G}_1(\omega)\tilde{G}_2^*(\omega) \\
S_{max}(\omega) &= S_{x_1 x_2}(\omega)\tilde{G}_1^*(\omega)\tilde{G}_2(\omega) \qquad (12)
\end{aligned}
$$

where $S_{x_1 x_2}(\omega)$ is the cross spectrum of the channel outputs. Then it holds:

$$
\begin{aligned}
arg\{S_{i_1 i_2}(\omega)\} &= \frac{1}{2}arg\{S_{min}(\omega)\} + \frac{1}{2}k_1\omega \\
arg\{S_{o_1 o_2}(\omega)\} &= \frac{1}{2}arg\{S_{max}(\omega)\} + \frac{1}{2}k_2\omega, \quad (13)
\end{aligned}
$$

where k_1, k_2 are integers, $S_{i_1 i_2}(\omega)$ is the cross-spectrum of the minimum-phase parts of $h_1(k)$ and $h_2(k)$, i.e., $i_1(k)$ and $i_2(k)$, and $S_{o_1 o_2}(\omega)$ is the cross-spectrum of the maximum-phase parts of $h_1(k)$ and $h_2(k)$, i.e., $o_1(k)$ and $o_2(k)$. Let $h_{min}(k)$, $h_{max}(k)$ be the inverse Fourier transforms of $S_{i_1 i_2}(\omega)$ and $S_{o_1 o_2}(\omega)$, respectively, i.e.,

$$
\begin{aligned}
h_{min}(k) &= i_1(k) * i_2^*(-k) \\
h_{max}(k) &= o_1(k) * o_2^*(-k) \qquad (14)
\end{aligned}
$$

The channel reconstruction is achieved after the following observation. The Fourier phase of $h_{min}(k)$ can be estimated from the data up to a linear phase component (via (12) (13)). The sequence $h_{min}(k)$, due to assumption on the channels, satisfies the conditions to be reconstructed from its phase only. Although a linear phase term is missing from its exact phase, $h_{min}(k)$ can still be reconstructed via the procedure outlined above. The reconstructed sequence $h_{min}(k)$, can be decomposed into its minimum and maximum phase parts, or equivalently, the minimum phase parts of the channels. This decomposition can be achieved in several ways (e.g. via cepstrum, or polynomial rooting). Similarly, $h_{max}(k)$ can be reconstructed from its phase given in (13) and lead to the maximum phase parts of the channels. Finally the channels can be reconstructed as the convolution of their minimum and maximum phase parts.

In the presence of noise and finite data lengths one can only obtain an estimate of $S_{min}(\omega)$ in (12), since errors will be present in the minimum-phase equivalent estimates. The propagation of noise to the phase of $S_{min}(\omega)$ and to the reconstructed sequence $h_{min}(k)$ is studied in [7], [6].

4. SIMULATION EXAMPLE

In this Section we demonstrate the robustness of the signal reconstruction from phase procedure, applied on the two channel problem introduced in Section 3. We show that although the phase is not exactly known, since noise is present in the observations (see (10)), and although the phase can only be estimated up to an unknown linear phase term, and the length of the channels is unknown, the proposed procedure still performs very well.

In order to simulate a single input two output situation, we created a channel as a superposition of two delayed raised cosine pulses,

$$h(t) = c(t, 0.11) + 0.8c(t - 0.7, 0.11) \qquad (15)$$

where $c(t, a)$ denotes a raised cosine pulse with roll-off factor a. The channel $h(t)$ was truncated to 6 symbol intervals and two channels of length 6 each were generated by sampling $h(t)$ twice during each period. A non-white random process $s(k)$ was also generated by filtering a white process through the filter $r(k) = [1, 0.6, 1.4]$. This process was then passed through the channels $h_1(k)$ and $h_2(k)$ to create $x_1(k)$ and $x_2(k)$, and noise at signal-to-noise ratio 10 dB was added to the outputs.

The channels were reconstructed according to the method of Section 3 and the signal reconstruction from phase procedure of Section 2 was used, with unknown linear phase term and unknown lengths for the sequences $h_{min}(k)$ and $h_{max}(k)$ (see (14)). We used 100 samples of the output sequences and carried out 100 Monte Carlo simulations, with different realizations for the input and noise processes. The reconstructed channels $h_1(k)$, $h_2(k)$ were combined after each simulation, to form an estimate of the channel $h(t)$. The results are shown in Fig. 1, where the actual channel $h(k)$ is shown in solid line, the average over 100 estimates is shown in dashed line and the dotted lines indicate standard deviation. We see that, even when the phase contains estimation errors, since the SNR is low, and even when the linear phase term and the length of the signals is unknown, the proposed procedure for system reconstruction is robust.

The performance of the proposed method depends critically on the phase samples used in forming the system (5). If these phase samples contain significant estimation errors, then the performance is degraded. However, as mentioned in Section 2, any set of $L > 2N - 1$ discrete phase samples can be used to form the system matrix Θ_x. The phase errors analysis in [6], [7] indicates which phase samples have low variance/bias and should be used to form Θ_x.

5. REFERENCES

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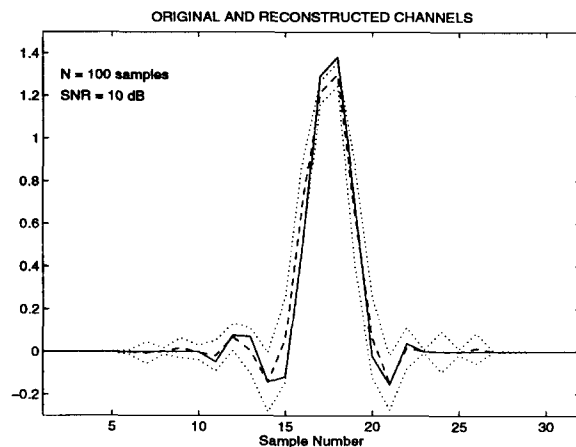
Figure 1: Proposed method: actual channel (solid line) and the average over 100 estimates (dashed line) at SNR=10 dB. Dotted lines indicate standard deviation. 100 output symbols were used for each estimate.

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