FIR FILTERS IN CONTINUOUS-TIME ENVELOPE CONSTRAINED FILTER DESIGN

B. Vo, T. Ho**, A. Cantoni* and V. Sreeram***

*Australian Telecommunications Research Institute,

Curtin University of Technology, Kent Street Bentley, W.A. 6102, Australia

**Electrical & Electronic Engineering Dept. University of Western Australia.

Email: vo@atri.curtin.edu.au

Abstract

Consider a continuous-time filter which in structure is comprised of an A/D converter, an FIR filter, a D/A converter and an analog post-filter. The envelope constrained (EC) filtering problem for this filter structure is to design it's digital component so as to minimize the effect of input noise whilst satisfying the constraint that the noiseless response of the filter to a specified excitation fits into a prescribed envelope. This problem is formulated as a quadratic programming (QP) problem with functional inequality constraints. Approximating this continuum of constraints by a finite set, the problem is solved by QP via active set strategy.

1. INTRODUCTION

Envelope constrained (EC) filter design is concerned with determining a filter so that its (noiseless) response to a specified input signal lies within a given envelope (see Figure 1) while minimizing the effect of input noise. A similar problem is the design of optimal pulse shape where the constraints are imposed on the impulse or frequency response of the filter [1].

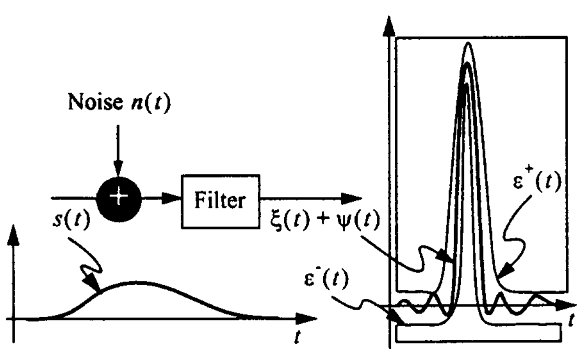

Figure 1: EC filter

In a variety of signal processing fields such as communication channel equalization, radar and sonar detection, robust antenna and filter design, seismology, EC filters are more directly relevant than least square filters [2].

Most works in this area primarily focus on a discrete-time version of the EC filtering problem. The primal-dual algorithm [2], [3] was an early approach used to solve the discrete-time EC filtering problem. More recently, the same problem was transformed into an unconstrained problem with continuously differentiable cost and solved by descent direction based algorithms [4].

Historically, a similar problem concerning the design of an optimal mismatched detection filter, subject to hard constraints on the output side-lobe had been formulated and reduced to an unconstrained problem in the space of regular Borel measures [5]. This continuous-time version of the EC filtering problem considered the case where the signals and impulse response of the optimal filter are finite in support. An approximate solution to this continuous-time problem was determined by discretizing the regular Borel measure space, thus reducing it to a finite dimensional problem. In [6], the finite dimensional problem was solved via steepest descent using directional Gateaux differentials. The exact solution to this problem is only of theoretical interest since it is unlikely that this optimal filter can be realized using existing circuitry without approximation. Suppose that we seek to realize an optimal EC filter by designing a network whose impulse response approximates that of the optimal EC filter in some sense. The important question then is "does this approximation still satisfy the envelope constraints?." We believe a better approach is to choose a particular realizable filter structure and then impose the envelope constraints to it.

By restricting the filter structure to a finite linear combination of orthonormal analog filters, the continuous-time EC filtering problem has been solved [7]. With advances in the development of digital processors, it is worthwhile to consider digital filter structures. In this paper, the design of a continuous-time EC filter with the principle component being an FIR filter is considered.

2. PROBLEM FORMULATION

This section formulates the EC filtering problem for an analog filter realized using digital techniques as shown in Figure 2. The resulting EC filtering problem is a finite dimensional optimization problem because the digital filter used is an FIR filter.

The post-filter functions as an interpolator to smooth the output of the Digital to Analog (D/A) block. In other words, the combined function of the D/A block and the post-filter is to interpolate the (discrete-time) output of the FIR filter. The output of the system varies from a staircase (no post-filtering or piece-wise constant interpolation) to much smoother waveforms depending on the type of post-filtering used. Another common and simple type of interpolation is linear interpolation [8]. In practice, the post-filter is often implemented as a lowpass filter (e.g. Butterworth, Chebyshev, elliptic and Bessel [9]) with cut-off at half of the sampling frequency.

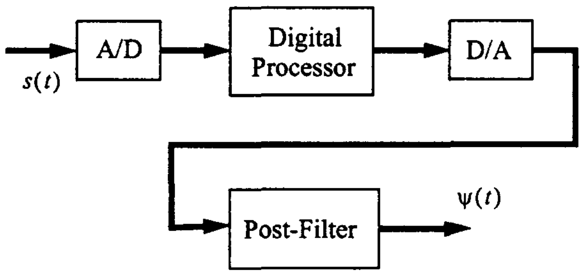

Figure 2: Configuration of digital realization

In the subsequent discussion, it is assumed that the incoming signal is sampled at or above the Nyquist rate. To simplify matters, the quantization errors inherent in the A/D process are neglected. This assumption ensures the linearity of the system, since quantization is a non-linear operation.

2.1 Filter Output

Let $\boldsymbol{u} = [u_0, \ldots, u_{n-1}]^T \in \boldsymbol{R}^n$ be vector of coefficients of the FIR filter and $h(t)$ be the impulse response of the post-filter. Then, the output of the digital processor when the system is excited by a continuous-time signal x defined on $\boldsymbol{R}$, is the sequence

$$\left\{\sum_{j=0}^{n-1} x[(i-j)\tau]u_j\right\}_{i=-\infty}^{\infty}$$

If x is square-integrable (i.e. $\{x(k\tau)\}_{k=-\infty}^{\infty}$ is square-summable), then the above sequence is bounded. This discrete-time signal then enters the D/A block and the resultant wave form can be written as

$$\sum_{i=-\infty}^{\infty} \sum_{j \in \hat{\Omega}} x[(i-j)\tau]u_j\Pi(t-i\tau), \qquad (1)$$

where Π is a pulse shape such that the sum in (1) converges for all $t \in [0, \infty)$ and the output of the D/A block is bounded and piece-wise continuous. An example of such pulse shape is a rectangular pulse of length τ, given by

$$\Pi(t) = \begin{cases} 1, t \in [0, \tau] \\ 0, \text{otherwise} \end{cases}.$$

In practice, Π has a shape that closely approximates the ideal rectangular pulse.

Let $h(t)$ denote the impulse response of the post-filter and $\Lambda(t) = \int_0^\infty \Pi(\lambda)h(t-\lambda)d\lambda$ the response of the post-filter to the pulse $\Pi(t)$. Then, the response of the hybrid filter to the continuous-time excitation x is given by

$$\psi(t) = \sum_{i=-\infty}^{\infty} \sum_{j \in \hat{\Omega}} x[(i-j)\tau]u_j\Lambda(t-i\tau), t \in [0, \infty) \qquad (2)$$

Since we are only dealing with excitation with support $[0, \infty)$, the index i in (2) can be taken from zero to infinity.

Assuming appropriate post-filtering such that (2) converges for all $t \in [0, \infty)$ (e.g. Bounded Input Bounded Output stability). Then, ψ is bounded and continuous on $[0, \infty)$. Suppose that the sampled input (finite support) is given by sequence $\{s(k\tau)\}_{k=0}^{m-1}$. Then, the output ψ is

$$\psi(t) = \sum_{i=0}^{N-1} \sum_{j=0}^{n-1} s[(i-j)\tau]u_j\Lambda(t-i\tau) \qquad (3)$$

where $N = n+m-1$.

Clearly, this sum converges for all t if the post-filter used is stable.

By defining

$$S = \begin{bmatrix} s(0) & 0 & \ldots & 0 \\ \vdots & s(0) & & \vdots \\ s((m-1)\tau) & \vdots & \ddots & 0 \\ 0 & s((m-1)\tau) & & s(0) \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & s((m-1)\tau) \end{bmatrix}_{N \times n}$$

the expression for ψ can also be expressed as

$$\psi(t) = [\Lambda(t), \ldots, \Lambda(t-(N-1)\tau)]S\boldsymbol{u} \equiv \boldsymbol{y}^T(t)\boldsymbol{u} \qquad (4)$$

2.2 Output Noise Power

A simple cost function we could use is the Euclidean norm of the discrete-time filter. However, the output noise power of this system is not directly proportional to this norm. In this section, an expression for the output noise power is derived, assuming stationary input noise samples.

Similar to the expression for the noiseless output, the output noise can be written as

$$\xi(t) = \sum_{l=-\infty}^{\infty} \sum_{j=0}^{n-1} n[(l-j)\tau]u_j\Lambda(t-l\tau)$$

making the change of variable $i = l-j$ yields

$$\xi(t) = \sum_{i=-\infty}^{\infty} n(i\tau)w^T(t-i\tau)u,$$

where

$$w(t) = [\Lambda(t), \ldots, \Lambda(t-(n-1)\tau)]^T.$$

By inspecting $E[\xi(t)]$ and $E[\xi(t+\lambda)\xi(t)]$, it is easily seen that the output noise is not stationary. However, if the input noise samples is a stationary process, it can be verified that the noise process ξ is cyclo-stationary with period τ, i.e. $E[\xi(t)]$ and $E[\xi(t+\lambda)\xi(t)]$ are periodic functions of t. The output noise power can be written as

$$E[\xi^2(t)] = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} R_{nn}((i-j)\tau)u^T w(t-i\tau)w^T(t-j\tau)u,$$

where $R_{nn}((i-j)\tau)$ is the auto correlation of the input noise samples.

Assuming that different input noise samples are independent and $E[n^2(i\tau)] = N_0$, the output noise power can be expressed as

$$E[\xi^2(t)] = N_0 \sum_{j=-\infty}^{\infty} (w^T(t-j\tau)u)^2.$$

Hence the averaged output noise power

$$\frac{1}{\tau}\int_0^\tau E[\xi^2(t)]dt$$

is independent of time and is a well-defined cost. Substituting for $E[\xi^2(t)]$, we obtain the cost function

$$f(u) = \frac{N_0}{\tau}\int_0^\tau \sum_{j=-\infty}^{\infty} u^T w(t-j\tau)w^T(t-j\tau)u\,dt.$$

By making the change of variable $\zeta = t-j\tau$, the cost function f becomes

$$f(u) = \frac{N_0}{\tau}\int_{-\infty}^{\infty} u^T w(\zeta)w^T(\zeta)u\,d\zeta = u^T L u. \quad (5)$$

where

$$L = \begin{bmatrix} L_0 & L_1 & \ldots & L_{n-2} & L_{n-1} \\ L_1 & L_0 & \ddots & & L_{n-2} \\ \vdots & & \ddots & & \vdots \\ L_{n-2} & & & L_0 & L_1 \\ L_{n-1} & L_{n-2} & \ldots & L_1 & L_0 \end{bmatrix},$$

$$L_j = \frac{N_0}{\tau}\int_{-\infty}^{\infty} \Lambda(\xi)\Lambda(\xi-j\tau)d\xi, \; j = 0, \ldots n-1.$$

Observe from (5) that the cost function is convex. Furthermore, $f(u) = 0$ implies that $w^T(\zeta)u = 0$ for almost all ζ, and since $w(\zeta) \neq 0$, it follows that $u = 0$. Hence f is strictly convex and the matrix L is positive definite.

2.3 Problem statement

The EC filtering problem for the structure of Figure 2 is then posed as

$$\begin{aligned} &\min u^T L u,\; u \in R^n \\ &\text{subject to } \varepsilon^-(t) \le y^T(t)u \le \varepsilon^+(t),\; \forall t \in \Omega \end{aligned} \quad (6)$$

where Ω is a compact interval on which the filter output must fit in the envelope.

This is a QP problem with a continuum of constraints. It can be easily verified that the constraints for this problem are convex. Hence, the EC filtering problem has a unique solution since the cost function is strictly convex.

This problem can be solved by approximating the continuum of constraints by a finite number of constraints as follows

$$\begin{aligned} &\min u^T L u,\; u \in R^n \\ &\text{subject to } \varepsilon^-(t_i) \le y^T(t_i)u \le \varepsilon^+(t_i),\; \forall i = 0, \ldots, M-1 \end{aligned}$$

Note that we only approximate the constraints of the primal problem as opposed to [6] which approximated the dual problem. Hence the constraints satisfaction at the instances $t_i, i = 0, \ldots, M-1$ can be guaranteed (if there exists a feasible solution). This approximate problem is a standard QP problem with a unique solution and can be efficiently solved by QP via active set strategy.

3. EXAMPLE

This example concerns the equalization of a digital transmission channel consisting of a coaxial cable operating at the DX3 rate (45Mb/sec) [10]. An equalizing filter is required to take the impulse response of a coaxial cable with a 30 dB loss at a normalized frequency of $1/\beta$ (β is the baud interval) as input and produces an output which lies within the envelope given by the DSX3 pulse template (see Figure 3).

With an FIR filter of 16 taps, two types of post-filters, namely a Butterworth and a Bessel each of order 5, are used. During the time interval of interest, $[0, 20\beta]$, the input signal is sampled at four times the baud rate i.e. every $\tau = \beta/4$ time units.

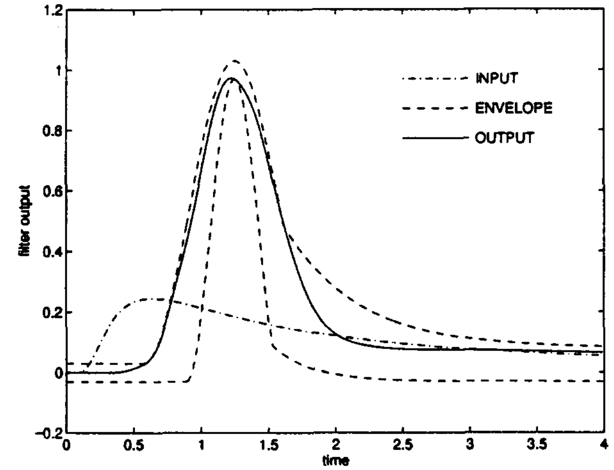

Figure 3: Butterworth Post-filter example.

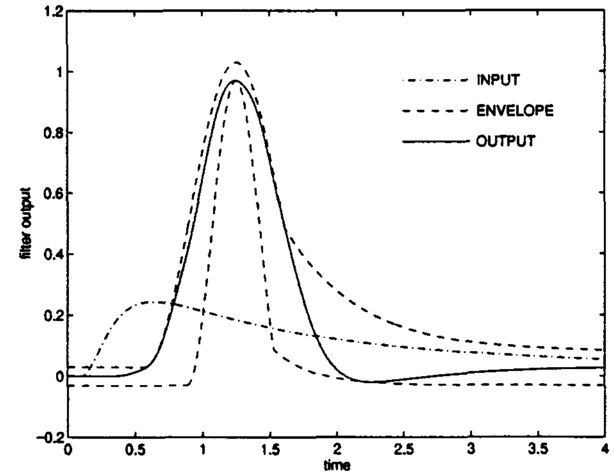

Figure 4: Bessel Post-filter example.

To obtain the approximate solution, we have constrained the output at $t_i = i\tau/8 = i\beta/32$, $i = 0, \ldots, 639$. For the Butterworth post-filter the optimal (or rather suboptimal) noise gain is 0.0146 and for the Bessel post-filter the suboptimal noise gain is 0.0182. Figures 3 and 4 show the input and output of these filters, the time axis are marked in units of the baud interval.

4. CONCLUSIONS

Proposed in this note is a continuous-time filter which consists of an A/D converter, an FIR filter, a D/A converter and an analog post-filter to solve the envelope constrained filtering problem. The problem has been formulated as a quadratic programming problem with functional inequality constraints. As an approximation we have solved the problem where the output is constrained at a finite number of points rather than an entire continuum. Numerical results have shown that useful solutions can be obtained.

References

[1] R. A. Nobakht and M. R. Civanlar, "Optimal pulse shape design for digital communication systems by projections onto convex sets", *IEEE Trans. Comm.*, **Vol. 43**, pp. 2874-2877, 1995.

[2] R. J. Evans, T. E. Fortmann, and A. Cantoni, "Envelope-constrained filter, Part I, theory and applications", *IEEE Trans. Information Theory*, **Vol. 23**, pp. 421-434, 1977.

[3] R. J. Evans, T. E. Fortmann, and A. Cantoni, "Envelope-constrained filter, Part II, adaptive structures", *IEEE Trans. Information Theory*, **Vol. 23**, pp. 435-444, 1977.

[4] W. X. Zheng, A. Cantoni, B. Vo and K. L. Teo, "Recursive procedures for constrained optimization problems and its application in signal processing", *IEE Proc. - Vision, Image and Signal Processing*, **Vol. 142**, No. 3, pp. 161-168, 1995.

[5] T. E. Fortmann, M. Athans, "Optimal filter design subject to output sidelobe constraints: Theoretical considerations", *Journal of Optimization Theory and Applications*, **Vol. 14**, No. 2, pp. 179-197, 1974.

[6] T. E. Fortmann, R. J. Evans, "Optimal filter design subject to output sidelobe constraints: Computational algorithm and numerical results", *Journal of Optimization Theory and Applications*, **Vol. 14**, No. 3, pp 271-290, 1974.

[7] B. Vo, Z. Zang, A. Cantoni and K. L. Teo, "Continuous-time envelope constrained filter design via orthonormal filters", *IEE Proc. - Vision, Image and Signal Processing*, **Vol. 142**, No. 6, pp. 389-394, 1995.

[8] B. Vo, A. Cantoni and K. L. Teo, "Envelope constrained filter with linear interpolator", *Technical Report*, SPL-TM006, ATRI, Curtin University of Technology, WA, 1995. Accepted for Publication in IEEE Trans. Signal processing.

[9] L. R. Rabiner and B. Gold, *Theory and application of digital signal processing*, Prentice-Hall, New Jersey, 1975.

[10] Bell Communications, "DSX-3 Isolated Pulse Template and Equations", *Technical Reference TR-TSY-000499*, pp 9-17, **Issue 2**, December 1988.