RECURSIVE ESTIMATION OF LINEARLY OR HARMONICALLY MODULATED FREQUENCIES OF MULTIPLE CISOIDS IN NOISE

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ABSTRACT

Recursive estimation of parameters of linearly or harmonically frequency modulated sinusoidal signals is considered. The algorithms simultaneously separate the measured signal to individual components and update signal parameters using estimated phase differencies. The main advantages of the proposed algorithms over standard tracking schemes such as e. g. the adaptive notch filter is zero asymptotic bias (zero tracking delay) in estimating of the instantaneous frequencies.

1. INTRODUCTION

Tracking of slowly varying parameters of multiple sinusoids or cisoids (complex-valued sinusoids) in additive noise is of great importance from both theoretical and practical points of view. It arises in many engineering applications such as radar, communications, control, biomedical engineering and others.

One of the most frequently studied algorithms for solving the above problem is the adaptive notch filter (ANF), [2]. Other methods include "multiple frequency tracker" (MFT), [7], an "adaptive (IIR) structure for separation, enhancement and tracking", [3], and "hyperstable adaptive line enhancer", [4]. Performance of these four algorithms is compared in [8]. The algorithms can be easily modified for tracking of similar parameters of real-valued signals, see the referencies therein.

All the algorithms mentioned above are well suited to the case when the time evolution of sinusoidal frequencies is either described by a random walk, or the frequencies are piecewise constant with jump changes. In some applications, however, the sinusoidal frequencies are piecewise linear or periodic functions of time. An example of such application is a coherent laser radar technology for remote sensing of vibrational characteristics of objects. In these cases, the frequency estimates provided by the above algorithms exhibit a nonzero tracking delay, which can be interpreted as an estimation bias. Elimination of this bias is the main intention of this paper.

The multicomponent signal under consideration, which is referred as the *carrier* in the sequel, is given as

$$y_n = \sum_{k=1}^{p} A_{kn} + v_n \qquad n = 0, 1, 2, \ldots \qquad (1)$$

where A_{kn} represents the k-th component at time instant n, v_n is the noise and p is the (known) number of the cisoids. The vector

$$A_n = (A_{1n}, \ldots, A_{pn})^T \qquad (2)$$

is called the *signal decomposition* vector. In (2), the superscript " T " denotes transpose. In the sequel, the superscript " H " denotes conjugate transpose.

The k-th instantaneous angular frequency is defined as the (backward) angle increment of A_{kn},

$$\omega_{kn} \triangleq \text{Arg}\left[\frac{A_{kn}}{A_{k,n-1}}\right] \qquad (3)$$

$k = 1, \ldots, p$. The sequence $\{\omega_{kn}\}$ is called the k-th *message*, for easy reference. The k-th instantaneous angular frequency rate-of-change is given as the (forward) one step increment of ω_{kn}, or equivalently,

$$\alpha_{kn} \triangleq \text{Arg}\left[\frac{A_{k,n+1}A_{k,n-1}}{A_{kn}^2}\right]. \qquad (4)$$

One should note that the stationary carrier is given by (1) and (3) with constant $\{\omega_{kn}\}$ and $\{|A_{kn}|\}$, so that $\{\alpha_{kn} \equiv 0\}$. The multiple linear FM signal is characterized by (1), (3) and (4) with constant $\{\alpha_{kn}\}$ and $\{|A_{kn}|\}$. The messages are linear functions of time in this case. Other definitions of the instantaneous frequency and frequency rate-of-change than those in (3) and (4) are also possible, [1]. Note that the instantaneous frequencies and frequency rates in the sense of the alternative definitions can be obtained by FIR filtering of the quantities in (3) and (4). [1]

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[1]The quantities in (3) and (4) are normalized to the sampling rate $f_s = 1[\text{Hz}]$.

The general multiple periodic (harmonic) FM signal is described by (1), (3) and

$$\omega_{kn} = \omega_{ckn} + \eta_{kn} \cos \phi_{kn} \qquad (5)$$

where ω_{ckn} is the mean value of the k-th message and η_{kn} and ϕ_{kn}, respectively, are the amplitude and instantaneous phase of the message. The angular frequency of the messages is given as the one-step increment of ϕ_{kn},

$$\nu_{kn} = \phi_{kn} - \phi_{k,n-1} \pmod{2\pi}. \qquad (6)$$

In the stationary case, ω_{ckn}, η_{kn} and ν_{kn} are independent of time. In this case, we can delete index n and write

$$\omega_{kn} = \omega_{ck} + \eta_k \cos(n\nu_k + \phi_{k0}). \qquad (7)$$

$k = 1, \ldots, p$, where ϕ_{k0} is the initial phase of the k-th message.

In Sections 2 and 3, algorithms for tracking of slowly varying parameters of linear FM signal and of periodic FM signal, respectively, are presented. Both algorithms are generalizations of the MFT algorithm, [7], and hence they are denoted by acronyms MFT-L and MFT-P. The main outcome of the algorithms is the messages, i.e. sequences $\{\hat{\omega}_{kn}\}$. As a by-product the algorithms give estimates of the signal decomposition vector A_n in (2) and of all other parameters. Initialization of the algorithms is discussed in Section 4 and numerical examples are presented in Section 5.

2. THE MFT-L ALGORITHM

The MFT-L algorithm, as well as MFT, proceeds at each time instant in two steps. The first step is a recursive update of the signal decomposition vector A_n and the second step consists of recursions for instantaneous frequencies and instantaneous frequency rates using estimated phase differences. The algorithm contains three user-chosen design variables, λ, ρ and μ lying in the interval (0,1).

The update formula for the vector $\hat{A}_n$ is given by the recursive minimizer of a quadratic loss function, that is the discounted sum of squared errors with a forgetting factor λ, cf. [11, p. 17]. For derivation of the formula it is assumed that the other parameters (ω_{kn} and α_{kn}) are constant and known. The derivation itself follows similar steps as those for the MFT, see Appendix A in [7], for further details see [10]. The result is

$$\hat{A}_n = \hat{F}_n \hat{A}_{n-1} + G_n^{-1} J \left[y_n - J^T \hat{F}_n \hat{A}_{n-1} \right] \qquad (8)$$

$$G_n = J J^T + \lambda \hat{F}_n G_{n-1} \hat{F}_n^H \qquad (9)$$

where

$$J = (1 \ldots 1)^T \qquad (p \times 1) \qquad (10)$$

$$\hat{F}_n = \text{diag}\left(e^{i\tilde{\omega}_{1n}}, \ldots, e^{i\tilde{\omega}_{pn}}\right) \qquad (11)$$

G_n is an auxiliary ($p \times p$) matrix, the symbol "diag" means a square diagonal matrix with the shown diagonal elements, "hats" denote estimated quantities, $i = \sqrt{-1}$, and $\tilde{\omega}_{kn}$ is the prediction of the k-th frequency at time instant n based on the data up to time $n-1$. In particular,

$$\tilde{\omega}_{kn} = \hat{\omega}_{k,n-1} + \hat{\alpha}_{k,n-1}. \qquad (12)$$

The second step consists in updating $\hat{\omega}_{kn}$ and $\hat{\alpha}_{kn}$ using the estimated phase differences, cf. the definitions (3) and (4),

$$\hat{\alpha}_{kn} = \mu \hat{\alpha}_{k,n-1} + (1 - \mu) \text{Arg}\left[\frac{\hat{A}_{kn} \hat{A}_{k,n-2}}{\hat{A}_{k,n-1}^2}\right] \qquad (13)$$

$$\hat{\omega}_{kn} = \rho(\hat{\omega}_{k,n-1} + \hat{\alpha}_{kn}) + (1 - \rho) \text{Arg}\left[\frac{\hat{A}_{kn}}{\hat{A}_{k,n-1}}\right] \qquad (14)$$

$k = 1, \ldots, p$. Note that the last term in (13) corresponds to estimating $\alpha_{k,n-1}$ rather than α_{kn} defined in (4). This discrepancy is unavoidable because no estimate of $A_{k,n+1}$ is available at time instant n. An alternative formulation of the recursions (13) and (14) is proposed, which appears to be less sensitive to wrong phase unwrapping, [7, 10]

$$\hat{\alpha}_{kn} = \hat{\alpha}_{k,n-1} + (1 - \mu) \text{Arg}\left[\frac{\hat{A}_{kn} \hat{A}_{k,n-2}}{\hat{A}_{k,n-1}^2 e^{i\hat{\alpha}_{k,n-1}}}\right] \qquad (15)$$

$$\hat{\omega}_{kn} = \hat{\omega}_{k,n-1} + \hat{\alpha}_{kn}$$
$$+ (1 - \rho) \text{Arg}\left[\frac{\hat{A}_{kn}}{\hat{A}_{k,n-1} e^{i(\hat{\omega}_{k,n-1} + \hat{\alpha}_{kn})}}\right] \qquad (16)$$

$k = 1, \ldots, p$. Different strategies for the choice of its design variables, λ, ρ and μ, are discussed in [10]. Note that for $\{\hat{\alpha}_{kn} \equiv 0\}$ the algorithm reduces to the MFT.

3. THE MFT-P ALGORITHM

The MFT-P algorithm employs the same decomposition A_n of the carrier as the MFT-L algorithm in (8)-(9) except for different definition of $\tilde{\omega}_{kn}$. This definition will be explored below. Also, this algorithm utilizes three user-chosen design variables, λ, ρ and μ lying in the interval (0,1).

Each component of the signal is characterized by a (3×1) state vector θ_{kn} equal to

$$\theta_{kn} = [\eta_{kn} \sin \phi_{kn}, \eta_{kn} \cos \phi_{kn}, \omega_{ckn}]^T. \qquad (17)$$

Once $\hat{\theta}_{kn}$ has been estimated $\hat{\eta}_{kn}$, $\hat{\phi}_{kn}$ and $\hat{\omega}_{kn}$ can be computed as

$$\hat{\eta}_{kn} = \sqrt{\hat{\theta}_{kn1}^2 + \hat{\theta}_{kn2}^2} \qquad (18)$$

$$\hat{\phi}_{kn} = \text{Arg}\left(\hat{\theta}_{kn2} + i\hat{\theta}_{kn1}\right) \qquad (19)$$

$$\hat{\omega}_{kn} = \hat{\theta}_{kn2} + \hat{\theta}_{kn3} \qquad (20)$$

where $\hat{\theta}_{kn\ell}$, $\ell = 1, 2, 3$ are components of $\hat{\theta}_{kn}$.

The main idea of the algorithm is to form preliminary estimates of the instantaneous frequencies as

$$\hat{\omega}_{kn}^{\text{prel}} = \text{Arg}\left[\frac{\hat{A}_{kn}}{\hat{A}_{k,n-1}}\right] \qquad (21)$$

and fit these estimates by a sinusoid-plus-constant model characterized by (17) using a specialized MFT algorithm. Note that a modification of the MFT algorithm for real-valued sinusoids was proposed in [9]. In particular, estimated state vectors $\hat{\theta}_{kn}$, $k = 1, \ldots, p$, are updated in parallel together with auxiliary 3×3 matrices $\{W_{kn}\}$ as

$$\hat{\theta}_{kn} = H_{k,n-1} \hat{\theta}_{k,n-1}$$

$$+W_{k,n}^{-1} J_3 \left[\hat{\omega}_{kn}^{\text{prel}} - J_3^T H_{k,n-1} \hat{\theta}_{k,n-1} \right] \quad (22)$$

$$W_{kn} = J_3 J_3^T + \rho H_{k,n-1} W_{k,n-1} H_{k,n-1}^T \quad (23)$$

where

$$H_{kn} = \begin{bmatrix} \cos \hat{\nu}_{kn} & \sin \hat{\nu}_{kn} & 0 \\ -\sin \hat{\nu}_{kn} & \cos \hat{\nu}_{kn} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (24)$$

$$J_3 = [0, 1, 1]^T . \quad (25)$$

Note that the structure of recursions (23), (22) is analogous to those in (8), (9). Finally, the instantaneous frequencies of the messages, $\hat{\nu}_{kn}$, are updated using the estimated angle increments of $\hat{\phi}_{kn}$, cf. (19), and forgetting factor μ,

$$\hat{\nu}_{kn} = \mu \hat{\nu}_{k,n-1} + (1-\mu)\text{Arg}\left[\frac{\hat{\theta}_{kn2} + i\hat{\theta}_{kn1}}{\hat{\theta}_{k,n-1,2} + i\hat{\theta}_{k,n-1,1}} \right] \quad (26)$$

The recursion (26) can be alternatively replaced by

$$\hat{\nu}_{kn} = \hat{\nu}_{k,n-1} + (1-\mu)\text{Arg}\left[\frac{\hat{\theta}_{kn2} + i\hat{\theta}_{kn1}}{\hat{\theta}_{k,n-1,2} + i\hat{\theta}_{k,n-1,1}} e^{-i\hat{\nu}_{k,n-1}} \right] \quad (27)$$

which eventually may be less sensitive to wrong phase unwrapping.

The one step ahead prediction of ω_{kn} which is needed in definition of $\tilde{F}_n$ in (11) is obtained as

$$\tilde{\omega}_{kn} = J_3^T H_{k,n-1} \hat{\theta}_{k,n-1} . \quad (28)$$

In summary, the MFT-P algorithm is given by (8), (9), (21)–(25), (27), (28). The problem of the choice of the design variables λ, ρ and μ still is to be investigated.

4. ALGORITHM INITIALIZATION

The proposed algorithms are not globally convergent in general and hence a proper initialization of the algorithms is essential. The initial transient of the tracking can be avoided or minimized if the algorithms are initialized by true or estimated signal parameters. This is, however, not always possible.

We propose to use the known superior global tracking properties of the MFT algorithm and the fact that MFT is a special case of both MFT-L and MFT-P. Thus for MFT-L it may be wise to force $\hat{\alpha}_{kn} = 0$ and $\mu = 1$ at the beginning of the tracking. Similarly, MFT-P reduces to MFT for $\nu_{kn} = 0$ and $\mu = 1$. (Note a minor technical problem with possibly singular matrix W_{kn}.) A reasonable way how to initialize the algorithm is to start by performing MFT until the mean square prediction error decreases and then switch to MFT-P.

In addition, in literature it is recommended for this kind of algorithms to begin the tracking with some small value of the forgetting factors and increase them recursively as e.g., [11]

$$\lambda_n = \lambda_c \lambda_{n-1} + (1 - \lambda_c)\lambda_\infty . \quad (29)$$

Here, λ_n, λ_∞ and λ_c are the n-th instantaneous design variable, the desired limit design variable and a constant that controls the speed of convergence of λ_n to λ_∞, respectively. The scheme (29) has been proved to shorten initial transients of many tracking procedures considerably.

5. NUMERICAL EXAMPLES

The tracking ability of the MFT-L algorithm is demonstrated in the following example. Noise-free data of length $N = 800$ that consists of two sinusoidal components with unit magnitudes is generated. The angular frequency of the first component is piecewise constant and equal to zero for $0 < n \leq 600$ and to one for $601 \leq n \leq 800$. The second frequency is piecewise linear, as shown in Figure 1 (dashed line). The signal is processed by the MFT-L algorithm with $\lambda = 0.9422$, $\rho = 0.9092$ and $\mu = 0.9782$. In addition, the first diagram in Figure 1 shows the result of application of MFT with $\lambda = \rho = 0.9604$ (dash-dotted line). The design variables of both procedures, MFT-L and MFT, are chosen in the way that the tracking ability of the procedures are approximately the same, according to analysis of the algorithms in [7, 10].
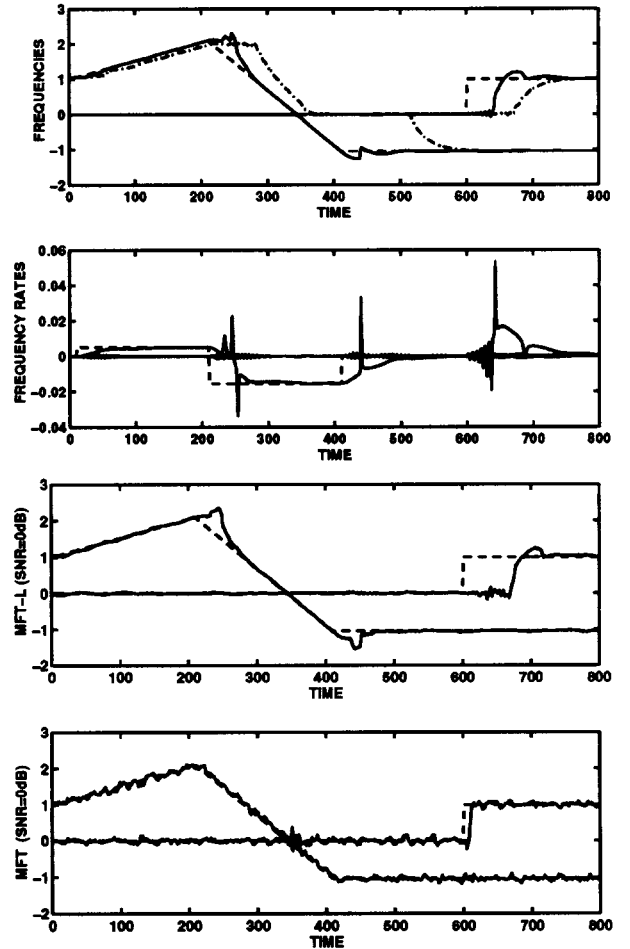

Figure 1: Performance of MFT-L (the first three diagrams) and performance of MFT (the first and the last diagram).

Both of the algorithms are initialized by aid of correct signal parameters. Note the extra tracking delay of the MFT algorithm at the place where the frequencies cross each other. This delay is avoided in the MFT-L algorithm. The lower two diagrams show the estimation in the same scenario but with additive noise. The SNR for each cisoids is 0dB. The third diagram shows the performance of MFT-

L, the fourth diagram is for MFT. Here, MFT has design variables $\lambda = \rho = 0.8$, in order to achieve visually similar tracking speed as that of MFT-L. Note the worse noise rejection of the MFT algorithm.

Performance of the MFT-P algorithm is tested on data of length $N = 800$ which consist of two sinusoidal components with unit magnitudes, again. The angular frequencies of both components are sinusoids-plus-constants with piecewise constant parameters, marked by dashed lines. The algorithm is initialized by aid of correct signal parameters. The first two diagrams show results for noise-free data, the latter two diagrams for data embedded in white Gaussian noise so that SNR is 10 dB. The design variables of MFT-P were $\lambda = 0.7$, $\rho = 0.85$ and $\mu = 0.975$. For comparison, the last diagram exhibits performance of the MFT algorithm with $\lambda = \rho = 0.9$.
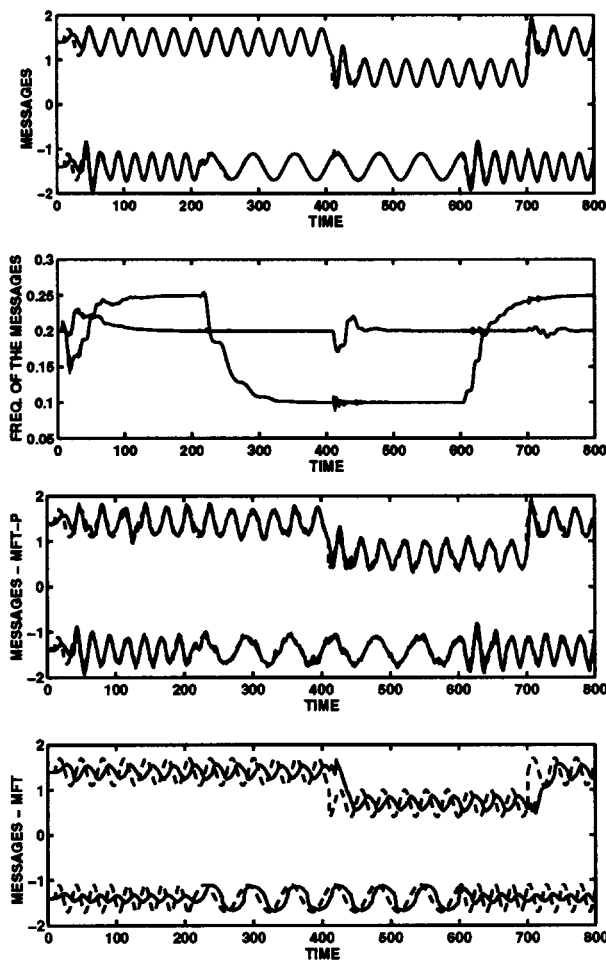

Figure 2: Performance of MFT-P (the first three diagrams) and performance of MFT (the last diagram).

We note that the tracking delay of MFT is a serious difficulty of this algorithm in tracking of rapidly varying frequencies even for relatively small forgetting factors λ and ρ, while MFT-P performs well in the given example even in the case of noisy data.

6. CONCLUSIONS

Two model-based variants of the MFT in [7] have been proposed, suitable for parametric adaptive estimation of linear (MFT-L) or periodic (MFT-P) variations of the instantaneous frequencies of a multi-component noisy carrier. Both variants comprise three scalar design variables for tuning the performance trade-off between noise rejection and tracking ability. Different strategies for the choice of design variables for MFT-L may be found in [10].

7. REFERENCES

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