BEST INPUT FOR OPTIMAL TRACKING RANDOMLY TIME-VARYING SYSTEMS : JUSTIFICATION OF ADAPTIVE PREDICTIVE STRUCTURE

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ABSTRACT

This paper presents a tracking analysis of the LMS algorithm used in order to identify system variations modeled by a random walk. We prove that the steady state properties are strongly related to the input characteristics. The input correlation degrades the performances. Consequently, best performances are obtained for white input. We justify then the coupled adaptive predictive structures with system identification in order to improve classical scheme steady state performances.

1. INTRODUCTION

The performance analysis of the LMS algorithm is concerned with transient state and steady state. For the estimation of unknown fixed filter, it is well known that the convergence speed is related to input power and is inversely proportional to the eigenvalue spread of the input correlation matrix. When the input is subject to power variations, algorithms such as Normalized LMS, Sign LMS (see for example [1][2]) improve transient performances. When the input is highly correlated, predictive structures are powerfull tools used to improve classical scheme transient properties ([3]).

In order to propose new adaptive schemes for identification of time-varying systems, it is necessary to analyse input influence on the tracking performances of the classical scheme. However, due to the difficulty of the steady state analysis, few algorithms are presented. In this paper, a tracking analysis of the LMS algorithm, used to identify random walk system variations is presented. From previous works, we analyse input power and input correlation effects on the algorithm performances : Normalized Excess Mean Square Error or misadjustement ($\mathcal{M}$) and Mean Square Deviation (η). However, with the classical approach, it is not possible to deduce the input correlation effect on the optimal tracking performances. The proposed approach which does not include the assumption of small step size (such as in [4]) allows us to study the input correlation effect on the algorithm performances.
Using this approach, we can deduce new adaptive structures in order to improve the classical identification scheme performances.

2. CLASSICAL APPROACHES : INPUT POWER INFLUENCE

2.1. Mathematical formulation

We consider a non stationary linear model which relates the two observed sequences of signal $x(k)$ and $y(k)$ according to $y(k) = F(k)^T X(k) + b(k)$; where $X(k) = [x(k), x(k-1), ..., x(k-N+1)]^T$ is the observation vector. The time variations of the unknown system $F(k)$ are modeled by random walk variations

$$F(k+1) = F(k) + \Omega(k+1) \quad (1)$$

where each component $\omega_i(k)$, $i = 0..N-1$ of $\Omega(k)$ is a centered sequence independent of the stationary input $x(k)$ and the additive output noise $b(k)$. We note $P_x = E\{x(k)^2\}$, $P_b = E\{b(k)^2\}$ and $P_\omega = E\{\omega_i(k)^2\}$, $\forall i = 0, .., N-1$
The adaptive system $H(k)$ that estimates $F(k)$ is governed by the LMS algorithm

$$H(k+1) = H(k) + \mu e(k) X(k) \quad (2)$$

where $e(k) = y(k) - H(k)^T X(k)$ is the estimation error and $\mu > 0$ is the step size.
With the standard formulation of the LMS using the deviation parameter $V(k) = H(k) - F(k)$ and with the

classical independence assumption between $X(k)$ and $V(k)$, we obtain

$$\mu P_x [R_x R_V + R_V R_x] = \mu^2 \Gamma + \mu^2 P_b P_x R_x + P_\omega R_\Omega \quad (3)$$

Where $R_x = E\{X(k)X(k)^T\}/P_x$ is the normalized input correlation matrix,
$R_\Omega = E\{\Omega(k)\Omega(k)^T\}/P_\omega$ is the normalized system variations correlation matrix, $R_V = E\{V(k)V(k)^T\}$ is the unknown covariance matrix of the deviation vector and $\Gamma = E\{X(k)X(k)^T R_V X(k)X(k)^T\}$.

Usually, for small step size, the term $\mu^2\Gamma$ is neglected, it yields to the following expressions (see [4]) of the Normalized Excess Mean Square Error (Misadjustement) $\mathcal{M} = (E\{e(k)^2\} - P_b)/P_b$ and of the Mean Square Deviation $\eta = E\{V(k)^T V(k)\}$

$$\mathcal{M}(\nu) = \frac{P_x Tr(R_x R_V)}{P_b} = \frac{1}{2}\left[\nu + \frac{N\delta}{\nu}\right] \quad (4)$$

$$\eta(\nu) = Tr(R_V) = \frac{P_b}{2P_x}\left[\nu + \frac{\delta Tr(R_x^{-1} R_\Omega)}{\nu}\right] \quad (5)$$

Where $\delta = NP_xP_\omega/P_b$ is the non stationary degree ([2]) and $\nu = \mu N P_x$ is the normalized step size.

2.2. Input Power Influence

For a given non stationary system, δ is proportional to the input power P_x. Equations (4 - 5) show that, as the input power increases as $\mathcal{M}$ increases and as η decreases. This is illustrated in figure 1 for a two-tap system. It is interesting to note that the tracking capabilities deduced from η and $\mathcal{M}$ are contradictory. The problem is that η describes properly tracking performances even though the only $\mathcal{M}$ is a measurable criterion.
In order to overcome this compromise and in order to improve tracking performances, Normalized LMS can can be used for randomly time-varying system identification when the input is subject to power variations ([5]).

3. PROPOSED APPROACH : INPUT CORRELATION INFLUENCE

3.1. Classical approaches Insufficiencies

Equation (5) shows that the input correlation affects η. When the components of $\Omega(k)$ are independent (R_Ω diagonal), $Tr(R_x^{-1}R_\Omega) \geq N$; the minimum value η_{min}

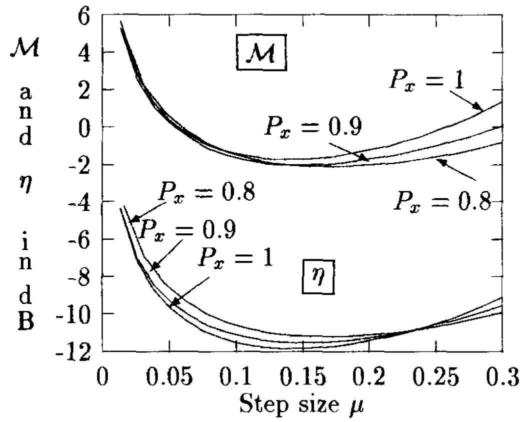


Figure 1: Input Power influence on $\mathcal{M}$ and η for $N = 2$, $P_b = 0.1$, independant components of $\Omega(k)$ ($P_\omega = 0.005$) and for a white input

of the Mean Square Deviation is then reached for white input

$$\eta \geq \eta^w = \frac{P_b}{2P_x}\left[\nu + \frac{N\delta}{\nu}\right] \quad (6)$$

Consequently, for small step size, white input leads to lower Mean Square Deviation η than for correlated input.
When the components of $\Omega(k)$ are correlated, we can demonstrate that, for a first order markovian input (a correlated input generated from a white noise), optimal performances are obtained for white input when the system variations are weakly correlated and for correlated input when the system variations are highly correlated ([6]).

However, from equation (4), the misadjustement $\mathcal{M}$ seems to not depend on the input correlation. This conclusion, true for small step size becomes false for optimal step size. We must then develop a specific approach in order to analyse the input correlation effect on the optimal tracking performances.

3.2. Proposed Approach

Unlike classical approaches, our approach does not use the assumption of small step size. For gaussian inputs, equation (3) is rewritten as

$$N[R_xR_V + R_VR_x] - 2\nu R_xR_VR_x - \nu R_x Tr(R_xR_V) \quad (7)$$
$$= \frac{P_b}{P_x}\left[\nu R_x + \frac{N\delta R_\Omega}{\nu}\right]$$

Analytical expression of R_V is difficult, however a set of linear equations (related to this matricial equation) can

be solved using symbolic-manipulation software package [6].
Note that the same approach was used in stationary context ($F(k) = F$) in order to determine the exact step size mean square stability bound for any filter length and any correlation statistics ([7]).

We illustrate our approach for a two-tap system ($N = 2$). The analytical expressions of $\mathcal{M}$ and η are

$$\mathcal{M}(\nu) = \frac{\left(\nu + \frac{2\delta}{\nu}\right) + \frac{\rho_x}{2-\nu}\left(\rho_x \nu^2 + 2\rho_\omega \delta\right)}{(2-2\nu) - \frac{2\rho_x^2\nu^2}{2-\nu}} \tag{8}$$

$$\eta(\nu) = \frac{P_b}{P_x}\frac{\left(\nu + \frac{2\delta}{\nu}\right) + \delta\frac{2-3\nu}{\nu(2-\nu)}\left(Tr\left(R_x^{-1}R_\Omega\right) - 2\right)}{(2-2\nu) - \frac{2\rho_x^2\nu^2}{2-\nu}} \tag{9}$$

Where $\rho_x = E\{x(k)x(k-1)\}/P_x$ and
$\rho_\omega = E\{\omega_1(k)\omega_2(k)\}/P_\omega$.
For white gaussian input ($\rho_x = 0$) or for small values of ν, we find again classical approach results ([2]).

Figure 2 corresponds to $\mathcal{M}$ and η for relatively rapid system variations ($\delta = 0.1$), with the input correlation coefficient $\rho_x = 0.5$ and $\rho_\omega = 0.5$. The proposed theoretical curve (1) is compared to the simulation curve (2). The curve (3) deduced from previous works, shows that the theory is in good agreement with the simulations for a small range of ν. With the proposed approach, the range of ν is enlarged and the optimum values of ν corresponding to minimum values of $\mathcal{M}$ and η are reached exactly.
For higher values of ν, the independence assumption between $X(k)$ and $V(k)$ limits the analysis.

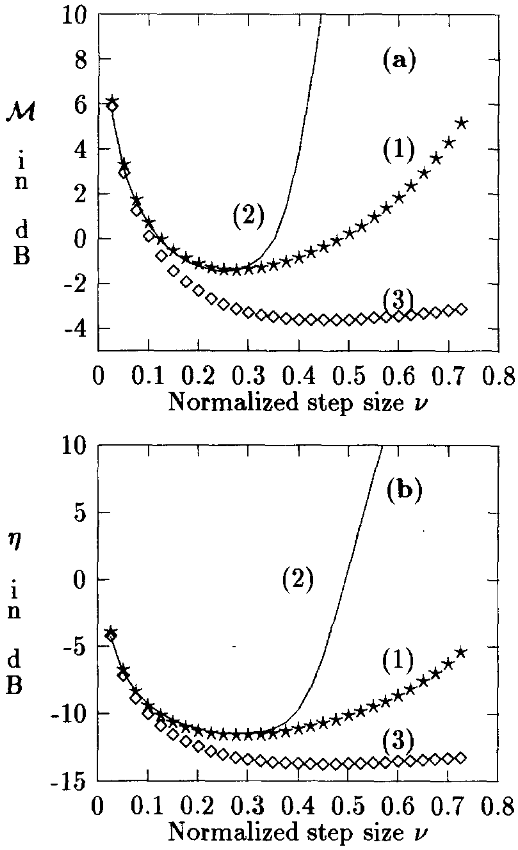


Figure 2: Comparison between the proposed approach ($\star$), the classical approach ($\diamond$) and the simulation results (-) for $P_b/P_x = 0.1$, $\delta = 0.1$, $\rho_x = 0.5$ and $\rho_\omega = 0.5$

4. JUSTIFICATION OF PREDICTIVE STRUCTURE

4.1. Best input choice for optimal $\mathcal{M}$

In this section, we propose bounding properties of $\mathcal{M}$ available for any, system length, system correlation and input correlation.
By taking the trace of both sides of equation (7) and using the following conditions $\lambda_{max} \geq 1$ and
$\lambda_{min}\frac{P_b}{P_x}\mathcal{M} \leq Tr(R_x R_V R_x) \leq \lambda_{max}\frac{P_b}{P_x}\mathcal{M}$, where λ_{min} (resp. λ_{max}) is the minimum (resp. the maximum) eigenvalue of the matrix R_x, we can deduce that

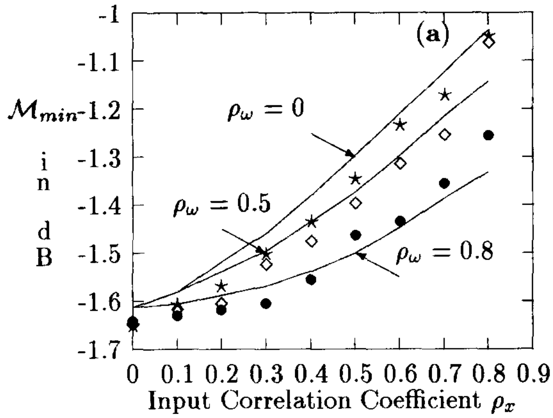
$$\mathcal{M} \geq \mathcal{M}^w = \frac{\nu + \frac{N\delta}{\nu}}{2 - \frac{2+N}{N}\nu} \tag{10}$$

The minimum value $\mathcal{M}^w$ of the misadjustement is reached for white input.

For a two-tap system, figure (3.a) shows the input correlation effect on the optimal value of $\mathcal{M}$ ($\mathcal{M}_{min}$) for three values of ρ_ω ($\rho_\omega = 0$, $\rho_\omega = 0.5$ and $\rho_\omega = 0.8$). As expected, figure (3.a) shows that, for any system correlation, the input correlation degradates the $\mathcal{M}_{min}$. A proposed solution to improve tracking performances when the input is correlated, is to apply an input prewhitening. In a recent work, we demonstrate that such structure, called "Adaptive Predictive Structure" improves significantly tracking performances ([8]).

4.2. Best input choice for optimal η

The exact determination of the input correlation influence on η is quite difficult for any step size and for any system characteristics. For a two-tap system, equation (5) shows that the input correlation and the system correlation appears in the term $Tr(R_x^{-1}R_\Omega)$. The conclusion announced in paragraph 3.1, for small step size,



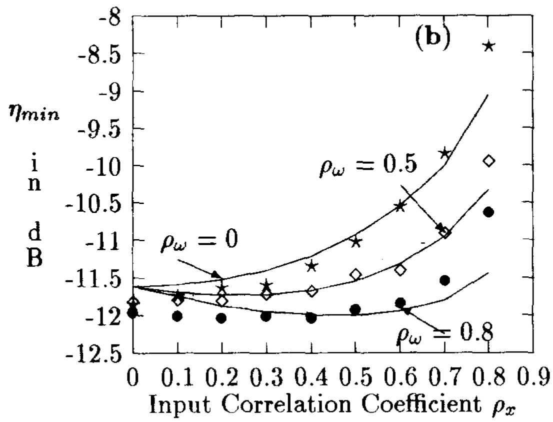


Figure 3: Input Correlation influence on the algorithm performances for optimal step size, theoretical result ('−'), simulation result ($\star$: for $\rho_\omega = 0$, $\diamond$: for $\rho_\omega = 0.5$ and $\bullet$: for $\rho_\omega = 0.8$)

is then generalized for optimal step size.

Figure (3.b) shows that, the optimal value of η (η_{min}) is obtained for white input when the system variations are weakly correlated ($\rho_\omega \leq 0.5$) and for correlated input when the system variations are highly correlated. The proposed solution to improve performances when the system is weakly correlated is to apply an input prewhitening as it is proposed for the misadjustement $\mathcal{M}$. However, for higher system correlation, the predictive structure is not adequate, wavelet transform based adaptive filtering is in that case a powerfull tool ([9]).

5. CONCLUSION

In this paper, we analyse the input characteristics influence on the tracking capabilities of the LMS algorithm in order to justify the use of adaptive predictive structure to improve classical scheme performances. To overcome the insufficiencies of the classical analysis, we develop a specific approach for high values of the step size. We prove that the input correlation degrades performances when the misadjustement $\mathcal{M}$ is the criterion used to measure the performances. This conclusion remains for weakly system variations correlation when the Mean Square Deviation η is analysed. This result justify the adaptive predictive structure where the adaptive system identification is coupled with an input prewhitening.

It is interesting to note that this conclusion related to steady state performance in a non stationary context is similar to the conclusion relative to transitory performance in a stationary context.

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