

STABILITY OF VARIABLE AND RANDOM STEPSIZE LMS

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ABSTRACT

The stability of variable stepsize LMS (VSLMS) algorithms with uncorrelated stationary Gaussian data is studied. It is found that when the stepsize is determined by the past data, the boundedness of the stepsize by the usual stability condition of fixed stepsize LMS is sufficient for the stability of VSLMS. When the stepsize is also related to the current data, the above constraint is no longer sufficient. Instead, both the upperbound and the lowerbound of the stepsize must be within a smaller region. An exact expression of the stability region is developed for single tap filter. The results are verified by computer simulations.

1. INTRODUCTION

There is a lot of interest recently in variable step-size least mean square (VSLMS) algorithms [1, 2, 3, 4]. The idea is to adjust the step size in a data dependent manner so as to improve the learning and tracking ability. Generally, the approach is to devise step size rules which give large steps when the estimated error is large and small steps when the error is small, thereby avoiding the tradeoff between convergence rate and misadjustment for fixed step size LMS. Of course, gradient algorithms (even those which perform line searches) are inferior to Newton algorithms for quadratic cost, and LMS is inferior to RLS (in terms of convergence rate for fixed misadjustment) on ill conditioned data with large eigenvalue spread in the input autocorrelation matrix. However, for many nonstationary data models, LMS is comparable to RLS (in terms of steady state tracking error). Significantly, in sufficiently noisy environment experiments show that properly designed VSLMS can approach RLS convergence rate and even exceed RLS tracking ability. Furthermore, the VSLMS have lower complexity and improved robustness (e.g., no divisions

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are required) than even the best designed fast least squares algorithms. These features of VSLMS suggest an important role for increasing performance in problems where fast tracking is required.

The weight updates in many of the VSLMS algorithms described in the literature take the following form:

$$\underline{w}(k+1) = \underline{w}(k) + \alpha_k e_k \underline{x}(k), \quad k = 0, 1, \dots$$

where α_k is the random step size and $\underline{x}(k)$ is the data vector and e_k is the error signal. Let

$$\begin{aligned} X(k) &= \{\underline{x}(k), \dots, \underline{x}(0)\} \text{ and} \\ E(k) &= \{e_k, \dots, e_0\} \end{aligned}$$

denote the data vectors and errors, respectively up until time k . The step size α_k in the literature is typically computed in one of two ways: (i) α_k depends on $X(k-1)$, $E(k-1)$, i.e., on the past data and errors; and (ii) α_k depends on $X(k)$, $E(k)$, i.e., on the current and past data and errors. Furthermore, a bound is enforced on α_k to ensure stability (presumably): $\underline{\alpha} \leq \alpha_k \leq \bar{\alpha}$.

The analysis of these VSLMS algorithms in the literature proceeds in two steps [2, 3]. First, a rigorous analysis of the mean square (MS) weight and MSE boundedness is made (or claimed), and second, an approximate analysis of the weight mean and covariance and MSE convergence is made and the steady state behavior is characterized (misadjustment, tracking error, etc.). The argument is made that at least one can guarantee stability (MS weight and MSE boundedness) rigorously, lending credence to the approximate analysis. Now for uncorrelated stationary Gaussian data and constant step size α it is known that a necessary and sufficient condition for MS weight and MSE boundedness is that [5]

$$0 \leq \alpha < \alpha^* \leq \frac{2}{3\text{Tr}\{R\}}$$

where $R = E\{\underline{x}(k)\underline{x}^T(k)\}$ (the exact value of α^* is determined by the eigenvalues of R ; see [5, Eqn. (30)]).

For the same data model but now with variable step size α_k , it is tacitly assumed that a sufficient condition for MS weight and MSE boundedness is that

$$0 \leq \underline{\alpha} \leq \alpha_k \leq \bar{\alpha} < \frac{2}{3\text{Tr}\{R\}}. \quad (1)$$

This assumption is straightforwardly correct if α_k is deterministic, but the usual case is where the α_k is data-dependent. One wonders, for example, if the gap between $\underline{\alpha}$ and $\bar{\alpha}$ plays a role in stability for such random step sizes.

In this paper we give necessary and sufficient conditions for the stability (MS weight and MSE boundedness) of VSLMS with uncorrelated stationary Gaussian data for a single tap filter. The key result is that Eq. (1) is a sufficient condition for MS weight and MSE boundedness if the step size is determined by the past data and error signals, but is not a sufficient condition if it additionally depends on the current data and error signal. We present an example of unstable behavior in the latter case when Eq. (1) is satisfied. The extension of these results to multiple tap filters will appear elsewhere (the results are similar to the single tap case except that only bounds on the stability region can be evaluated). This work is a fundamental generalization of the results in [5] for stability of fixed step size LMS algorithms, and is important for rigorously establishing (at least under strong classical assumptions on the data) the stability of a class of adaptive algorithms which are of increasing interest in practical applications. It complements the approximate analysis of the VSLMS in [2, 3] which was shown to have value for the prediction of steady state behavior.

2. THE MAIN RESULT

Consider the single tap VSLMS algorithm

$$\begin{aligned} w(k+1) &= w(k) + \alpha_k e_k x(k) \\ e_k &= (w^* - w(k))x(k) + n_k \end{aligned}$$

for $k = 0, 1, \dots$. We shall make the independence and Gaussian assumptions as in [5], namely $x(k)$ is white Gaussian with mean 0 and variance σ_x^2 , n_k is white Gaussian with mean 0 and variance σ_n^2 , and $\{x(k)\}, \{n_k\}$ are independent.

Define the *MSE stability region*

$$S = \{(\underline{\alpha}, \bar{\alpha}) : \sup_k E\{e_k^2\} < \infty \text{ for all } \{\alpha_k\} \text{ such that } \underline{\alpha} \leq \alpha_k \leq \bar{\alpha} \text{ w.p. } 1\}.$$

We note that we can replace $E\{e_k^2\}$ by $E\{w^2(k)\}$ and get the same set S , at least if α_k depends at most on

$X(k), E(k)$ (if α_k depends on the future values of $x(i), e_i, i > k$, this is not true in general). From [5] for the fixed step size case $\alpha_k = \alpha$ it is known that the MSE is bounded if and only if $0 \leq \alpha < \alpha^* = \frac{2}{3\sigma_x^2}$. It follows that $S \subset \{(\underline{\alpha}, \bar{\alpha}) : 0 \leq \underline{\alpha} \leq \bar{\alpha} < \alpha^*\}$. Now define the *lower envelope* of the MSE stability region

$$\underline{\alpha}(\bar{\alpha}) = \inf\{\underline{\alpha} : (\underline{\alpha}, \bar{\alpha}) \in S\}, \quad 0 \leq \bar{\alpha} < \alpha^*,$$

$(\underline{\alpha}(\bar{\alpha}), \bar{\alpha})$ is well-defined since $(\underline{\alpha}, \bar{\alpha}) \in S$. Observe that

- if $0 \leq \bar{\alpha} < \alpha^*$ and $\underline{\alpha}(\bar{\alpha}) < \underline{\alpha}$, then $E\{e_k^2\}$ is bounded for all $\{\alpha_k\}$ such that $\underline{\alpha} \leq \alpha_k \leq \bar{\alpha}$ w.p. 1.
- if $0 \leq \bar{\alpha} < \alpha^*$ and $\underline{\alpha} < \underline{\alpha}(\bar{\alpha})$, then $E\{e_k^2\}$ is unbounded for some $\{\alpha_k\}$ such that $\underline{\alpha} \leq \alpha_k \leq \bar{\alpha}$ w.p. 1.

We have the following two results

Case 1. If α_k depends on $X(k-1), E(k-1)$, then

$$S = \{(\underline{\alpha}, \bar{\alpha}) : 0 \leq \underline{\alpha} \leq \bar{\alpha} < \alpha^*\}.$$

Case 2. If α_k depends on $X(k), E(k)$, then

$$S = \{(\underline{\alpha}, \bar{\alpha}) : 0 \leq \underline{\alpha} \leq \bar{\alpha} < \alpha^*, \rho(\underline{\alpha}, \bar{\alpha}) < 1\}.$$

where

$$\begin{aligned} \rho(\underline{\alpha}, \bar{\alpha}) &= 1 - 2\bar{\alpha}\sigma_x^2 + 3\bar{\alpha}^2\sigma_x^4 \\ &\quad + 4\sigma_x^2(\bar{\alpha} - \underline{\alpha})\Phi_1\left(\frac{1}{2p\sigma_x^2}\right) \\ &\quad - 4\sigma_x^4(\bar{\alpha}^2 - \underline{\alpha}^2)\Phi_2\left(\frac{1}{2p\sigma_x^2}\right) \\ \Phi_i(x) &= \frac{2}{\sqrt{\pi}} \int_0^x t^{2i} e^{-t^2} dt \end{aligned}$$

and $p = (\underline{\alpha} + \bar{\alpha})/2$ (the $\Phi_i, i \geq 1$, can be expressed in terms of the probability integral Φ_0). Furthermore, if $\rho(\underline{\alpha}, \bar{\alpha}) \geq 1$ then an example which has unbounded MSE is

$$\alpha_k = \begin{cases} \bar{\alpha} & \text{if } px^2(k) > 1 \\ \underline{\alpha} & \text{if } px^2(k) < 1 \end{cases}. \quad (2)$$

2.1. An Example

In Fig. 1 we plot the the lower envelope of the stability region (the region lies above the envelope and below the line $\underline{\alpha} = \bar{\alpha}$) for random step size which depend on current and past data and error (Case 2 above). Here $\sigma_x^2 = 1$ and consequently $\alpha^* = 2/3$. We compare this result with simulation of VSLMS for $w^* = 1, \sigma_n^2 = 0.01$, and the step size given in Eq. (2). We set $\bar{\alpha}$ to $\frac{1}{3}$ so that from Fig. 1 the smallest value of $\underline{\alpha}$ for MSE

boundedness is about 0.016. In Figs. 2 and 3 we show the MSE learning curve (averaged over 100 trials) for VSLMS with $\underline{\alpha} = 0.0032$ and $\bar{\alpha} = 0.08$, respectively. We also show the MSE learning curve of LMS with $\alpha = \frac{1}{3}$. The MSE of VSLMS in Fig. 2 appears to diverge, while that in Fig. 3 appears to be bounded, in accord with the theory.

3. REFERENCES

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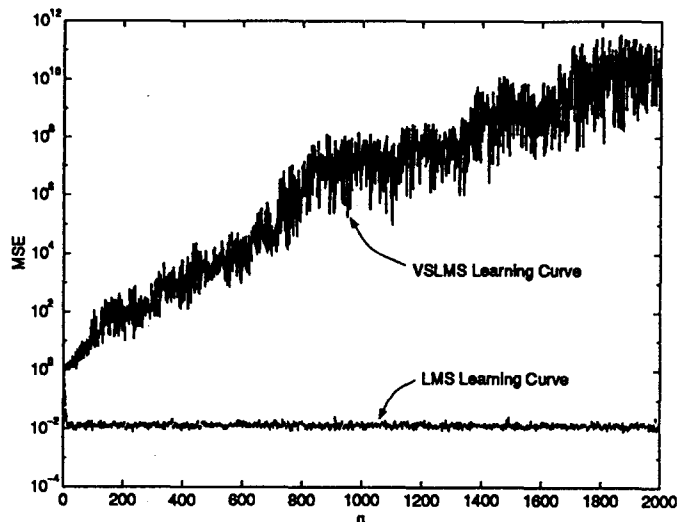


Figure 2: MSE learning curves for VSLMS with $\underline{\alpha} = 0.0032$ and $\bar{\alpha} = \frac{1}{3}$ and LMS with $\alpha = \frac{1}{3}$.

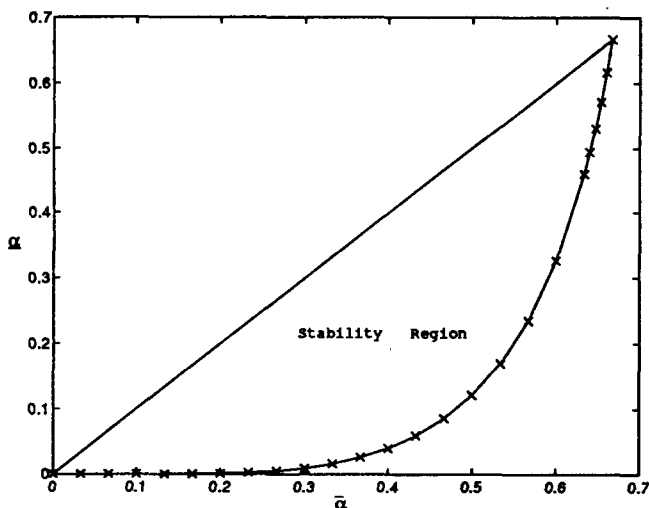


Figure 1: Stability region of random stepsize LMS.

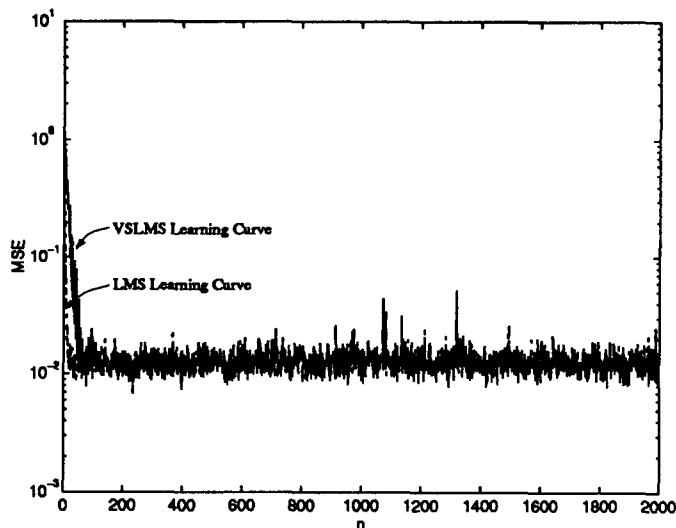


Figure 3: MSE learning curves for VSLMS with $\underline{\alpha} = 0.08$ and $\bar{\alpha} = \frac{1}{3}$ and LMS with $\alpha = \frac{1}{3}$.

