ON THE RECURSIVE TOTAL LEAST-SQUARES

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ABSTRACT

In this paper, by exploiting the Total Least-Square (TLS) closed-form solution and using state-space structure in Krein space, we will show that the solution of the TLS problems can be computed via the recursive Kalman filtering algorithm. This makes it possible to use the TLS for real-time applications.

1. INTRODUCTION

The method of least-square (LS) has been applied in variety of problems in science and engineering, since it plays a central role in studying the inconsistency of the overdetermined system of equations $\mathbf{y} \sim \mathbf{Hx}$, where $\mathbf{H}$ is a NxM matrix with N > M, $\mathbf{y}$ is a Nx1 and $\mathbf{x}$ is a Mx1 vector, respectively. In standard approach, the LS formulation implicitly assumes the $\mathbf{y}$ is the only term that is subjected to disturbances, that is, $\mathbf{y} + \mathbf{v} = \mathbf{Hx}$. However, in many situations, such as Spectral Estimation, Infinite-Impulse Response (IIR) Adaptive filtering, one is faced with the problem in which both $\mathbf{H}$ and $\mathbf{y}$ are subjected to noise. And for that reason, the Total Least Squares (TLS), or Errors-In-Variables has received increasing attention due to its robustness of handling this situation.

The main tool to compute the solution of the TLS is the well-known Singular Value Decomposition (SVD) [1], which is not suited for parallel computation and can not be used for real-time applications due to expensive computational cost ($O(N^3)$) and inherently non-recursive operations. In this paper, by exploiting the TLS closed-form solution using state-space structure in Krein space, we will show that the solution of the TLS problems can be computed via the Kalman filter algorithm. And since Kalman Filter technique is well-studied for the last three decades, many results can be directly applied for the recursive TLS solution, including square-root and parallel implementations, especially it can be used for real-time applications.

2. FORMULATION

2.1 Basic TLS and Its Solution

The solution of the TLS has been known for sometimes, but it is often credited to Golub and Van Loan for their modern treatment, and their development of computationally efficient and numerically reliable algorithm [1]. The computational aspects and new algorithms of TLS are well-studied and documented by Van Huffel [2]. Adopting from [2], Figure 1 shows the difference between the LS and the TLS methods. In the LS technique, the estimated quantity is obtained by projecting the measurements to the space that is spanned by $\mathbf{H}$ (white-surface in Figure 1), while in the TLS case, the solution is obtained by projecting the measurement to an appropriated space that is spanned by the estimate of $\mathbf{H}$ (crossed-surface in Figure 1).

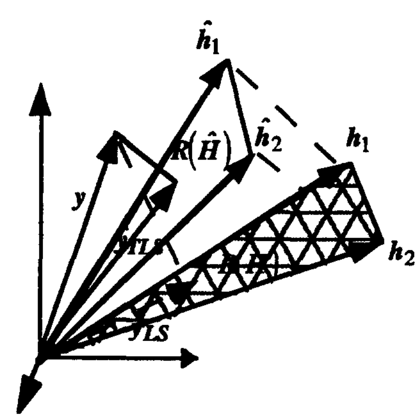

Figure 1: The difference between LS and TLS.

The following review section follows closely from the presentation in [2]. Mathematically, the solutions of LS and TLS are formulated as,

LS:

$$\min_{y \in R^N} \left\| y - \hat{y}_{LS} \right\|$$
subject to $\hat{y}_{LS} \in R(H)$

TLS:

$$\min_{[\hat{H}, \hat{y}_{TLS}] \in R^{Nx(M+1)}} \left\| [H, y] - [\hat{H}, \hat{y}_{TLS}] \right\|_F$$
subject to $\hat{y}_{TLS} \in R(\hat{H})$

It can be easily shown [2] that the solution of the TLS problem has the following form,

$$\hat{x} = \left(H^T H - \sigma_{M+1}^2 I\right)^{-1} H^T y \qquad (1)$$

Note that, if the minimum singular value is zero, then the solution of the TLS and LS problems are identical.

2.2 Recursive TLS (RTLS) Solution

By exploiting the state-space structures in Krein space,

$$\begin{aligned} \mathbf{x}_n &= \mathbf{F}_n \mathbf{x}_{n-1} + \mathbf{G}_n \mathbf{u}_n \\ \mathbf{y}_n &= \mathbf{H}_n \mathbf{x}_n + \mathbf{v}_n \end{aligned} \qquad (2)$$

with appropriated settings on the known Gramian matrices of $\{\mathbf{u}_n, \mathbf{v}_n, \mathbf{x}_0\}$,

$$< \begin{bmatrix} \mathbf{u}_n \\ \mathbf{v}_n \\ \mathbf{x}_0 \end{bmatrix}, \begin{bmatrix} \mathbf{u}_m \\ \mathbf{v}_m \\ \mathbf{x}_0 \end{bmatrix} >_K = \begin{bmatrix} \mathbf{Q}_n \delta_{nm} & 0 & 0 \\ 0 & \mathbf{R}_n \delta_{nm} & 0 \\ 0 & 0 & \Pi_0 \end{bmatrix}$$

In [3], it has been shown that the state estimate of (2) via Kalman filtering algorithm is equivalent to solving the following two minimization problems with indefinite quadratic forms,

Deterministic Least-Squares Problem:

$$\min_x \left[x^T \Pi^{-1} x^T + (y - Hx)^T W^{-1} (y - Hx) \right] \qquad (3)$$

Stochastic Least-Squares Problem:

$$\min_K \left\{ \Pi - KH\Pi - \Pi H^* K^* + K\left[H \Pi H^* + W\right] K^* \right\} \quad (4)$$

The solution of (3) and (4) can also be shown to have an identical form of,

$$\hat{x} = \left[\Pi^{-1} + H^* W^{-1} H\right]^{-1} H^* W^{-1} y \qquad (5)$$

Comparing (2) and (3), we can conclude that the solution of (2) can be computed by setting **W** as the covariance of the measurement noise and Π as the covariance of the state **x** in the stochastic least-squares problem with the following state-space structure,

$$\begin{aligned} \mathbf{x}_n &= \mathbf{x}_{n-1} \\ \mathbf{y}_n &= \mathbf{H}_n \mathbf{x}_n + \mathbf{v}_n \end{aligned} \qquad (6)$$

We can also recognize that the solution of the TLS (1) is same as the solution of these minimization problems with the following settings,

$$\Pi = -\sigma_{M+1}^2 I \text{ and } W = I \qquad (7a)$$

or,

$$\Pi = -I \text{ and } W = \sigma_{M+1}^2 I \qquad (7b)$$

The first setting, equation (7a), has been suggested by [3]. Under this setting, the state vector will have negative "covariance" matrix. The recursive construction of the solution of the TLS problem can be obtained by initializing the Kalman filter algorithm using (7a). If one knows precisely the minimum singular value of [**H y**], then the estimation of the state vector will be identical to the TLS solution. If an approximated value of the minimum singular value of [**H y**] is used, then one obtains an approximation of the TLS solution. However, if the approximated value is greater than the true minimum singular value, the algorithm will diverge because the coefficient matrix of (3) is negative-definite . We are proposing the second setting, equation (7b). Under this setting, **W** plays the role of the covariance of the measurement noise in the stochastic least-squares problem, σ_{N+1}, the minimum singular value of the partition matrix [**H y**], can be interpreted as the "standard deviation" of the white, measurement noise. This situation occurs frequently in using Kalman filtering algorithm with unknown measurement noise.

In those situations, we have to initialize the measurement noise variance with an arbitrary positive constant and estimate it by some means. In the problem at hand, we estimate this variance σ_{M+1}, by using the Incremental Condition Estimation (ICE) algorithm presented in [4], which costs O(M) operations at each measurement update.

In addition, if the data are known (batch mode), we can use ICE to estimate the minimum signular value of partition matrix [H y], then using Kalman filtering algorithm with setting (7a) to compute the recursive solution.

2.3 Applications

The TLS is applicable in solving many signal processing problems where the LS has been applied in the past. In this paper, we choose spectral estimation as a specific application due to its popularity. To estimate unknown frequencies, we set up a Forward (or a Forward-Backward) Linear Prediction equation,

$$\begin{bmatrix} y(0) & \cdots & y(M-1) \\ \vdots & \ddots & \vdots \\ y(N-M-1) & \cdots & y(N-1) \end{bmatrix} \begin{bmatrix} x(1) \\ \vdots \\ x(M) \end{bmatrix} = \begin{bmatrix} y(M) \\ \vdots \\ y(N) \end{bmatrix}$$

In this case, the appended data matrix has a form of,

$$[\mathbf{H}\ \mathbf{y}] = \begin{bmatrix} y(0) & \cdots & y(M) \\ \vdots & \ddots & \vdots \\ y(N-M-1) & \cdots & y(N) \end{bmatrix}$$

Denote the row vector of the data matrix at time n as $\mathbf{a}_n$ and using QR to update the data matrix,

$$\mathbf{a}_n^T = [y(n-M) \ \cdots \ \cdots \ y(n-1)]$$

$$\begin{bmatrix} \mathbf{R}_n \\ \mathbf{0} \end{bmatrix} = \mathbf{Q}_n \begin{bmatrix} \mathbf{R}_{n-1} \\ \mathbf{a}_n^T \ y(n) \end{bmatrix}$$

Using ICE [4], we can estimate the minimum singular value of $\mathbf{R}_n$ and the coefficients can be computed via Kalman filtering algorithm,

$$\hat{\mathbf{x}}_n = \hat{\mathbf{x}}_{n-1} + \frac{\mathbf{P}_n \mathbf{a}_n^T}{\sigma_{M+1}^2 + \mathbf{a}_n \mathbf{P}_n \mathbf{a}_n^T} \left[y(n) - \mathbf{a}_n \hat{\mathbf{x}}_{n-1} \right]$$

$$\mathbf{P}_n = \mathbf{P}_{n-1} + \frac{\mathbf{P}_n \mathbf{a}_n^T \mathbf{a}_n \mathbf{P}_n}{\sigma_{M+1}^2 + \mathbf{a}_n \mathbf{P}_n \mathbf{a}_n^T}, \quad \mathbf{P}_0 = -\mathbf{I}$$

3. RESULTS

Two experiments were conducted using the RTLS algorithm for the problem of spectral estimation. Two sinusoids are simulated with normalized frequencies f_1=.2 and f_2=.25. The number of data, N, is 40 and the dimension of the weight vector, M, is 5 with varied signal-to-noise ratio of 20 dB and 30 dB. In the first experiment, we used the setting of equation (5a) with different values of the approximated minimum singular values σ_{N+1} as shown in Figure 2. This result confirmed two important points: (1) The solution of the TLS solution can be computed exactly by Kalman Filter if the minimum singular value of σ_{N+1} is known, and (2), the approximation solution is quite sensitive to the approximated minimum singular value. In the second experiment, we used the setting of (5b) and the ICE algorithm to estimate the minimum singular value at each time update. The results are shown in Figure 3. It can be seen that the solution of the TLS problem using the SVD and Kalman filter algorithm are almost identical but the computational cost is reduced greatly using RTLS, in order of $O(2NM^2)$, as shown in Figure 4.

4. CONCLUSION

In this paper, by exploiting the TLS closed-form solution using state-space structure in Krein space, we have shown that the solution of the TLS problems can be computed via the Kalman filtering algorithm. This RTLS alogorithm composes of two independent steps. The first step is to compute the minimum singular value via ICE, which served as noise variance estimator. The second step is Kalman filtering with updated measurement noise variance. Both QR and Kalman filtering techniques are well-studied in literature. This makes it possible to apply the algorithm for real-time applications.

5. REFERENCES

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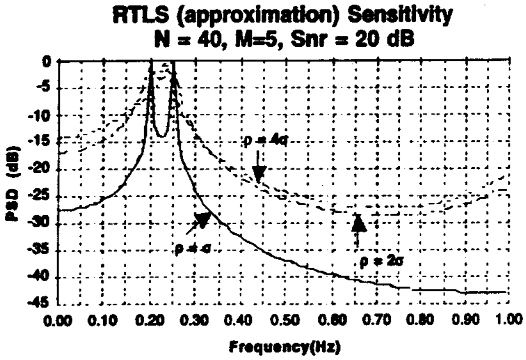
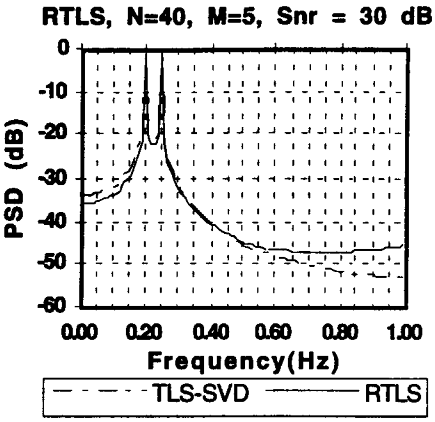
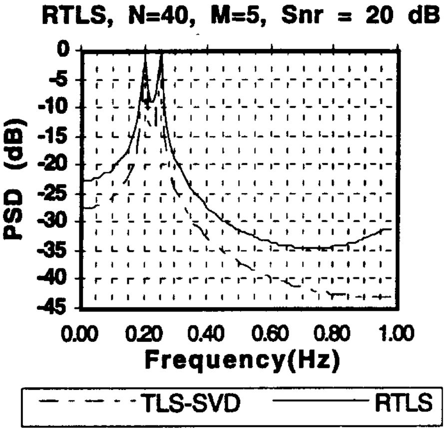


Figure 2. Sensitivity of approximated minimum singular value for setting (7a)



(a)



(b)

Figure 3. Spectral Estimation using TLS. a) SNR = 30 dB. b) SNR = 20 dB.

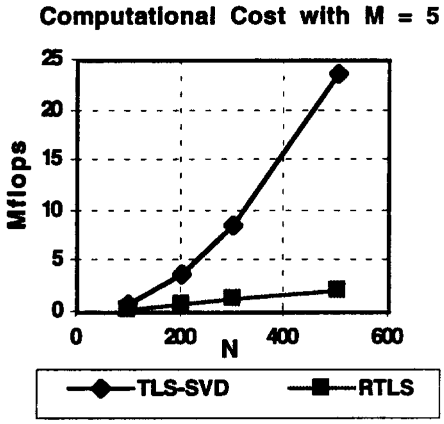


Figure 4. Computational Costs