LOCALIZED SUBCLASSES OF QUADRATIC TIME-FREQUENCY REPRESENTATIONS*

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ABSTRACT

We discuss the existence of classes of quadratic time-frequency representations (QTFRs), e.g. Cohen, power, and generalized time-shift covariant, that satisfy a time-frequency (TF) concentration property. This important property yields perfect QTFR concentration along group delay curves. It also (1) simplifies the QTFR formulation and property kernel constraints as the kernel reduces from 2-D to 1-D, (2) reduces the QTFR computational complexity, and (3) yields simplified design algorithms. We derive the intersection of Cohen's class with the new power exponential class, and show that it belongs to Cohen's localized-kernel subclass. In addition to the TF shift covariance and concentration properties, these intersection QTFRs preserve power exponential time shifts, important for analyzing signals passing through exponentially dispersive systems.

1. INTRODUCTION

Quadratic time-frequency representations (QTFRs) can be grouped into classes based upon covariance properties that they satisfy. Cohen's class QTFRs [1, 2, 3] are covariant to constant time-frequency (TF) shifts, affine class QTFRs [4, 2, 3] are covariant to dilations and constant time shifts, and the hyperbolic class [5, 6] and power classes QTFRs [7, 5] are covariant to dilations and dispersive time shifts.

In this paper, we focus on subclasses of QTFRs whose kernels have a localized structure. Such QTFRs satisfy a TF concentration property, an important property that yields perfect concentration of the QTFRs along specified group delay curves for certain types of signals. This concentration property of a QTFR $T_{X_c}(t,f)$ of a signal $x_c(t)$ (with Fourier transform $X_c(f)$) is given by

$$X_c(f)=r(f)e^{-j2\pi c\lambda(\frac{f}{f_r})} \Rightarrow T_{X_c}(t,f)=r^2(f)\delta(t-c\mu(f)). \quad (1)$$

Thus, given the signal $X_c(f)$ with amplitude function $r(f)\geq 0$ and one-to-one phase function $\lambda(b)\in\mathbb{R}$, we want the QTFR to be perfectly concentrated along the signal's group delay $c\mu(f)=c\frac{d}{df}\lambda(\frac{f}{f_r})$. Here, $f_r>0$ is a fixed reference frequency. This property results in a simplification of the signal-independent 2-D kernel to a 1-D localized-kernel characterizing the QTFRs. The localized-kernel idea was introduced by the Bertrands based upon tomography [8], and then extended by others to the affine localized-kernel subclass [4, 9, 10], and to the hyperbolic localized-kernel subclass [6, 5]. In this paper, we investigate the subclass of QTFRs that satisfy the TF concentration property in (1) for other known QTFR classes such as Cohen's class [1, 2, 3] and the power classes [7, 5].

In addition, we consider classes of QTFRs that satisfy the generalized time-shift covariance property defined as

$$(\mathcal{D}_c^{(\xi)}X)(f)=e^{-j2\pi c\,\xi(\frac{f}{f_r})}X(f)\Rightarrow$$
$$T_{\mathcal{D}_c^{(\xi)}X}(t,f)=T_X(t-c\,\tau(f),f) \quad (2)$$

with $\tau(f)=\frac{d}{df}\xi(\frac{f}{f_r})$ and $\xi(b)$ a one-to-one function [11, 12]. We derive the concentration property kernel constraints for these generalized time-shift covariant classes, thus providing a unifying framework for many localized-kernel QTFRs.

2. COHEN's LOCALIZED-KERNEL QTFRs

Any Cohen's class [1] QTFR, $T_X^{(C)}(t,f)$, with signal-independent kernel, satisfies the time-shift covariance property given by (2) with $\xi(b)=b$ and $\tau(f)=1/f_r$, and the frequency-shift covariance property defined as

$$(\mathcal{M}_\nu X)(f)=X(f-\nu)\Rightarrow T^{(C)}_{\mathcal{M}_\nu X}(t,f)=T_X^{(C)}(t,f-\nu). \quad (3)$$

These shift covariances are important in applications where the TF structure is analyzed with constant TF resolution such as in speech. Based on the TF shift covariance properties, any member of Cohen's class can be expressed in terms of a 2-D kernel $\Phi_T^{(C)}(\hat{f},\nu)$ [2, 3]

$$T_X^{(C)}(t,f)=\int\int\Phi_T^{(C)}(f-\hat{f},\nu)\,U_X(\hat{f},\nu)\,e^{j2\pi t\nu}d\hat{f}d\nu \quad (4)$$

where[1] $U_X(\hat{f},\nu)=X(\hat{f}+\nu/2)X^*(\hat{f}-\nu/2)$. Two interesting Cohen's class QTFRs are the Wigner distribution (WD) and Cohen's counterpart of the κth power Bertrand distribution [5] (CBD_κ, $\kappa\neq 0$). These QTFRs are defined as in (4) with well-localized kernels given by $\Phi_{\mathrm{WD}}^{(C)}(\hat{f},\nu)=\delta(\hat{f})$ and $\Phi^{(C)}_{\mathrm{CBD}_\kappa}(\hat{f},\nu)=\delta\big(\hat{f}-\frac{f_r}{\kappa}\ln\big[\sinh(\frac{\kappa\nu}{2f_r})/\frac{\kappa\nu}{2f_r}\big]\big)$; they result in highly concentrated representations for linear and exponential chirps, respectively.

We propose *Cohen's localized-kernel subclass* (LCC) as the subclass of QTFRs consisting of all Cohen's class QTFRs whose kernel $\Phi_T^{(C)}(\hat{f},\nu)$ in (4) is perfectly localized along a curve $\hat{f}=F_T^{(C)}(\nu)$ in the $(\hat{f},\nu)$-plane, i.e.

$$\Phi_T^{(C)}(\hat{f},\nu)=G_T^{(C)}(\nu)\,\delta(\hat{f}-F_T^{(C)}(\nu)). \quad (5)$$

The 2-D kernel of each LCC QTFR is reduced to two, 1-D functions, $F_T^{(C)}(\nu)\in\mathbb{R}$ and $G_T^{(C)}(\nu)\geq 0$, that uniquely characterize the QTFR. This greatly simplifies the QTFR analysis as it reduces the LCC formulation in (4) to $T_X^{(C)}(t,f)=\int G_T^{(C)}(\nu)\,U_X(f-F_T^{(C)}(\nu),\nu)\,e^{j2\pi t\nu}d\nu$, i.e. from two integrals to just one integral. Furthermore, it simplifies the kernel constraints for desirable QTFR properties [2].

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[1] Unless otherwise specified, integration limits are $-\infty:\infty$.

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The LCC is also important as it contains all Cohen's class QTFRs that satisfy the TF concentration property in (1). In particular, we show next that the localized-kernel structure in (5) is necessary for the corresponding QTFR to satisfy the TF concentration property along group delay curves in (1) [9]. For a given $\tau(f)$ and $\lambda(b)$ in (1), and if $\Lambda_{f,\nu}^{(C)}(\hat{f}) \triangleq \lambda((f-\hat{f}+\nu/2)/f_r) - \lambda((f-\hat{f}-\nu/2)/f_r)$ is assumed to be one-to-one and differentiable for fixed f,ν, then (1) is satisfied by a Cohen's class QTFR if and only if the following three conditions are satisfied:

C–I There exists a real function $F_T^{(C)}(\nu)$, independent of f, that satisfies $\Lambda_{f,\nu}^{(C)}(F_T^{(C)}(\nu))=\nu\mu(f)$ for all $f,\nu\in\mathbb{R}$.

C–II The ratio $\tau^2(f)/R_{f,\nu}^{(C)}(\hat{f})$ is independent of f, where $R_{f,\nu}^{(C)}(\hat{f}) \triangleq \tau(f-\hat{f}+\nu/2)\,\tau(f-\hat{f}-\nu/2)$.

C–III The kernel $\Phi_T^{(C)}(\hat{f},\nu)$ of the LCC is given by (5) with $G_T^{(C)}(\nu)=\tau^2(f)/R_{f,\nu}^{(C)}(F_T^{(C)}(\nu)) \geq 0$.

In particular, fixing $\tau(f)=e^{\frac{\kappa f}{2f_r}}$, $\kappa\neq 0$, condition C–II always holds with $G_T^{(C)}(\nu)=e^{\frac{\kappa}{f_r}F_T^{(C)}(\nu)}$. For example, if $\lambda(b)=b$ in property (1), condition C–I is satisfied for arbitrary $F_T^{(C)}(\nu)$. If $\lambda(b)=e^{\kappa b}$, then the TF concentration property for power exponential chirps $X_c(f)$ in (1) is satisfied by Cohen's class QTFRs with $F_T^{(C)}(\nu)=\frac{f_r}{\kappa}\ln[\sinh(\frac{\kappa\nu}{2f_r})/\frac{\kappa\nu}{2f_r}]$ and $G_T^{(C)}(\nu)=e^{\frac{\kappa}{f_r}F_T^{(C)}(\nu)}=\sinh(\frac{\kappa\nu}{2f_r})/\frac{\kappa\nu}{2f_r}$ in (5).

Some LCC QTFRs include the Wigner distribution (with 1-D kernels in (5) given by $G_{\mathrm{WD}}^{(C)}(\nu)=1$ and $F_{\mathrm{WD}}^{(C)}(\nu)=0$) and the CBD_κ (with 1-D kernels in (5) given by $G_{\mathrm{CBD}_\kappa}^{(C)}(\nu)=1$ and $F_{\mathrm{CBD}_\kappa}^{(C)}(\nu)=\frac{f_r}{\kappa}\ln[\sinh(\frac{\kappa\nu}{2f_r})/\frac{\kappa\nu}{2f_r}]$). Next, we propose another important localized-kernel subclass member, Cohen's intersection with the *new κth power exponential class*.

2.1. Intersection with Power Exponential Classes

The κth power exponential class is obtained by exponentially warping [12] the κth power class [7, 5]. Any member of this new class, $T_X^{(E\kappa)}(t,f)$, $\kappa\neq 0$, satisfies two covariance properties: the frequency-shift covariance in (3), and the power exponential time-shift covariance (defined as in (2) with $\xi(b)=e^{\kappa b}$ and $\tau(f)=\frac{\kappa}{f_r}e^{\kappa\frac{f}{f_r}}$), which is an important property for analyzing signals passing through exponentially dispersive systems. Based on these two covariance properties, any power exponential QTFR can be written as $T_X^{(E\kappa)}(t,f)=\frac{|\kappa|}{f_r}\int\int\Gamma_T^{(E\kappa)}(\frac{f_1-f}{f_r},\frac{f_2-f}{f_r})X(f_1)X^*(f_2)\,e^{j2\pi t\frac{f_r}{\kappa}\left[e^{\frac{\kappa}{f_r}(f_1-f)}-e^{\frac{\kappa}{f_r}(f_2-f)}\right]}df_1df_2$ where $\Gamma_T^{(E\kappa)}(b_1,b_2)$ is a 2-D kernel characterizing $T^{(E\kappa)}$. An important power exponential QTFR is the CBD_κ. When κ=1, the power exponential class simplifies to the exponential class [12].

If we constrain any member of Cohen's class in (4) to satisfy the power exponential time-shift covariance, the resulting constraint on Cohen's class kernel is given by $\Phi_T^{(C)}(\hat{f},\nu)=G_T^{(C)}(\nu)\delta(\hat{f}-F_{\mathrm{CBD}_\kappa}^{(C)}(\nu))$. Note that this is in the form of the localized-kernel in (5) with $G_T^{(C)}(\nu)$ arbitrary, but $F_T^{(C)}(\nu)=F_{\mathrm{CBD}_\kappa}^{(C)}(\nu)=\frac{f_r}{\kappa}\ln[\sinh(\frac{\kappa\nu}{2f_r})/\frac{\kappa\nu}{2f_r}]$ fixed to the $F_T^{(C)}(\nu)$ function of the CBD_κ. Thus, the intersection between Cohen's and the power exponential class is a subclass of the LCC. Any intersection member, $T_X^{(C\cap E\kappa)}(t,f)$, such as the CBD_κ, only needs the 1-D kernel $G_T^{(C)}(\nu)$ to characterize it. Also, any $T_X^{(C\cap E\kappa)}(t,f)$ is a smoothed CBD_κ,

$$T_X^{(C\cap E\kappa)}(t,f)=\int g_T^{(C)}(t-\hat{t}\,)\,\mathrm{CBD}_\kappa(\hat{t},f)\,d\hat{t}, \qquad (6)$$

where $g_T^{(C)}(t)$ is the inverse Fourier transform of $G_T^{(C)}(\nu)$ in (5). For κ=1, the intersection in (6) simplifies to another member of the LCC, the intersection of Cohen's class with the exponential class [12, 13]. The intersection in (6) can also be obtained by exponentially warping the intersection of the κth power class with the hyperbolic class [5, 6].

3. POWER LOCALIZED-KERNEL QTFRs

The κth power class consists of QTFRs that satisfy the scale covariance and the power time-shift covariance properties [7, 5]. The power time-shift covariance is defined as in (2) with $\xi(b)=\xi_\kappa(b)=\mathrm{sgn}(b)|b|^\kappa$ and $\tau(f)=\tau_\kappa(f)=\frac{d}{df}\xi_\kappa(\frac{f}{f_r})$, and it is important for multiresolution analysis of signals propagating through power dispersive systems. Based on these two covariances, any power QTFR, $T_X^{(P\kappa)}(t,f)$, $\kappa\neq 0$, can be expressed as

$$T_X^{(P\kappa)}(t,f)=\frac{1}{|\xi_\kappa(\frac{f}{f_r})|}\int\int\Phi_T^{(P)}\Big(\frac{-b}{\xi_\kappa(\frac{f}{f_r})},\frac{\beta}{\xi_\kappa(\frac{f}{f_r})}\Big)V_X^{(\kappa)}(b,\beta)\cdot e^{j2\pi\frac{t}{\tau_\kappa(f)}\beta}\,db\,d\beta \qquad (7)$$

where $V_X^{(\kappa)}(b,\beta)=f_rU_{\mathcal{W}_\kappa X}(f_rb,f_r\beta)$ is the product in (4) of the power warped [7] signal $(\mathcal{W}_\kappa X)(f)$. The 2-D kernel $\Phi_T^{(P)}(b,\beta)$ uniquely characterizes any power QTFR $T^{(P\kappa)}$. The κth power class can also be obtained by power warping the affine class [7, 5].

The *κth power localized-kernel subclass* (LPC_κ) consists of all power QTFRs that also satisfy the TF concentration property (1). This subclass provides the generalization of the affine localized-kernel subclass [9, 10] since for κ=1 the power class reduces to the affine class. The LPC_κ consists of all power QTFRs whose kernel $\Phi_T^{(P)}(b,\beta)$ in (7) is perfectly localized along a curve $b=F_T^{(P)}(\beta)$ in the (b,β)-plane,

$$\Phi_T^{(P)}(b,\beta)=G_T^{(P)}(\beta)\,\delta(b-F_T^{(P)}(\beta)), \qquad (8)$$

where $F_T^{(P)}(\beta)\in\mathbb{R}$ and $G_T^{(P)}(\beta)\geq 0$ are 1-D functions that characterize the QTFR $T^{(P\kappa)}$. When the reduced kernel in (8) is inserted in (7), it simplifies the formulation of LPC_κ QTFRs. It also simplifies the property kernel constraints for the power classes [5] to be in terms of two 1-D functions.

Power QTFRs satisfy the TF concentration property (1) if and only if their kernel has the structure in (8) resulting from three conditions listed below. If the functions $\tau(f)$ and $\lambda(b)$ in (1) are given, if $\Lambda_{f,\beta}^{(P)}(b)\triangleq\lambda(\frac{f}{f_r}\xi_{\frac{1}{\kappa}}(-b+\frac{\beta}{2}))-\lambda(\frac{f}{f_r}\xi_{\frac{1}{\kappa}}(-b-\frac{\beta}{2}))$ (for $\xi_\kappa(b)$ in (7)) is assumed one-to-one and differentiable for fixed f,β, and if $\kappa\neq 0$, then

P–I There exists a function $F_T^{(P)}(\beta)$, independent of f, that satisfies $\Lambda_{f,\beta}^{(P)}(F_T^{(P)}(\beta))=\frac{f}{\kappa}\mu(f)\beta$ for all $f,\beta\in\mathbb{R}$.

P–II The ratio $\tau^2(f)/R_{f,\beta}^{(P)}(b)$ is independent of f where $R_{f,\beta}^{(P)}(b)=|b^2-\frac{\beta^2}{4}|^{\frac{1-\kappa}{2\kappa}}\tau(f\xi_{\frac{1}{\kappa}}(-b+\frac{\beta}{2}))\tau(f\xi_{\frac{1}{\kappa}}(-b-\frac{\beta}{2}))$.

P–III The kernel $\Phi_T^{(P)}(b,\beta)$ of the LPC_κ is given by (8) with $G_T^{(P)}(\beta)=\tau^2(f)/R_{f,\beta}^{(P)}(F_T^{(P)}(\beta))\geq 0$.

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Class	$\Lambda^{(\text{class})}_{f,\beta}(b)$	$R^{(\text{class})}_{f,\beta}(b)/f_r\tau(f)$	Condition G–I	Condition G–II	Condition G–III
GC or GA	$\lambda(\Xi^{(\text{class})}_{f,\beta}(b)) - \lambda(\Xi^{(\text{class})}_{f,-\beta}(b))$	$r(f_r\Xi^{(\text{class})}_{f,\beta}(b))r(f_r\Xi^{(\text{class})}_{f,-\beta}(b)) \cdot \lvert\xi'(\Xi^{(\text{class})}_{f,\beta}(b))\xi'(\Xi^{(\text{class})}_{f,-\beta}(b))\rvert^{-\frac{1}{2}}$	$\Lambda^{(\text{class})}_{f,\beta}(F^{(\text{class})}_T(\beta)) = \Upsilon^{(\text{class})}(f)\,\beta$	$G^{(\text{class})}_T(\beta) = \frac{r^2(f)}{R^{(\text{class})}_{f,\beta}(F^{(\text{class})}_T(\beta))}$	$\Phi^{(\text{class})}_T(b,\beta) = G^{(\text{class})}_T(\beta)\delta(b - F^{(\text{class})}_T(\beta))$

Table 1: *Necessary conditions for generalized warped Cohen's (class = GC) or generalized warped affine (class = GA) QTFRs to satisfy the TF concentration property in (1). Here,* $\Xi^{(GC)}_{f,\beta}(b)=\xi^{-1}(\xi(\frac{f}{f_r})-b+\frac{\beta}{2})$ *and* $\Xi^{(GA)}_{f,\beta}(b)=\xi^{-1}(\xi(\frac{f}{f_r})(-b+\frac{\beta}{2}))$ *where* $\xi(b)$ *is given in (9) and (10). Also,* $\Upsilon^{(GC)}(f)=\mu(f)/\tau(f)$ *and* $\Upsilon^{(GA)}(f)=\xi(\frac{f}{f_r})\,\mu(f)/\tau(f)$.

Condition P–II is always satisfied if $r(f)=r_0|f|^\alpha$, $r_0>0$, $\alpha\in\mathbb{R}$, in which case $G^{(P)}_T(\beta)=|(F^{(P)}_T(\beta))^2-\frac{\beta^2}{4}|^{\frac{\kappa-1-2\alpha}{2\kappa}}$. For a given $\lambda(b)$, one must find an $F^{(P)}_T(\beta)$ to satisfy condition P–I. In particular, if $\lambda(b)=\xi_\kappa(b)$, then LPC$_\kappa$ QTFRs satisfy (1) for arbitrary $F^{(P)}_T(\beta)$ and fixed $G^{(P)}_T(\beta)$ as above. If $\lambda(b)=\ln b$, then P–I is satisfied for $F^{(P)}_T(\beta)=-\frac{\beta}{2}\coth(\frac{\beta}{2})$ and $G^{(P)}_T(\beta)$ simplifies to $(\frac{\beta/2}{\sinh(\beta/2)})^{\frac{\kappa-1-2\alpha}{\kappa}}$.

Some QTFRs of the LPC$_\kappa$ include the power Wigner distribution, $W^{(\kappa)}_X(t,f)$, with $G^{(P)}_{W^{(\kappa)}}(\beta)=-F^{(P)}_{W^{(\kappa)}}(\beta)=1$, and the power Bertrand distribution, $P^{(\kappa)}_0(t,f)$, with 1-D kernels $G^{(P)}_{P^{(\kappa)}_0}(\beta)=\frac{\beta}{2}/\sinh(\frac{\beta}{2})$ and $F^{(P)}_{P^{(\kappa)}_0}(\beta)=-\frac{\beta}{2}\coth(\frac{\beta}{2})$. An important subclass of the LPC$_\kappa$ is the intersection between the κth power and hyperbolic classes [5, 6]. Any power QTFR that belongs in the intersection also satisfies the hyperbolic time-shift covariance property (defined as in (2) with $\xi(b)=\ln b$ and $\tau(f)=1/f$), and its kernel in (7) is given by $\Phi^{(P)}_T(b,\beta)=G^{(P)}_T(\beta)\delta(b+\frac{\beta}{2}\coth(\frac{\beta}{2}))$. The intersection kernel has the localized-kernel form (8) with $G^{(P)}_T(\beta)$ arbitrary, but $F^{(P)}_T(\beta)=F^{(P)}_{P^{(\kappa)}_0}(\beta)=-\frac{\beta}{2}\coth(\frac{\beta}{2})$ fixed to that of the power Bertrand distribution. Thus, intersection QTFRs also satisfy the TF concentration property (1). As the intersection QTFRs satisfy at least four desirable properties, the QTFR formulation is simplified even further, and the number of possible applications is increased.

4. GENERALIZED WARPED LOCALIZED-KERNEL QTFRs

Generalized time-shift covariant QTFRs satisfy the generalized time-shift covariance property in (2) for given $\xi(b)$ and $\tau(f)$ [11, 12]. These QTFRs are especially useful when the shift is matched to group delay functions $\tau(f)$, and they are important for analyzing signals passing through systems with matched group delay characteristics. Two QTFR classes that satisfy (2) are the generalized warped Cohen's class and the generalized warped affine class [11, 12, 14]. These two classes depend on the choice of $\xi(b)$ in (2) that defines the type of warping applied to either Cohen's class or to the affine class. Thus, the generalized warped classes provide a unifying framework for Cohen's, affine, hyperbolic, power, exponential, and power exponential classes [12].

QTFRs of the generalized warped Cohen's class (GC), $T^{(GC)}_X(t,f)$, are obtained by warping Cohen's class QTFRs [12, 14] using a mapping specified by $\xi(b)$. Any GC QTFR can be expressed in terms of a 2-D kernel $\Phi^{(GC)}_T(b,\beta)$ as

$$T^{(GC)}_X(t,f)=\iint \Phi^{(GC)}_T(\xi(\frac{f}{f_r})-b,\beta)V_X(b,\beta)e^{j2\pi\frac{t\beta}{\tau(f)}}\,db\,d\beta \quad (9)$$

where $V_X(b,\beta)=f_rU_{\mathcal{W}_\xi X}(f_rb,f_r\beta)$ and the warped signal is $(\mathcal{W}_\xi X)(f)=X(f_r\xi^{-1}(\frac{f}{f_r}))/|\xi'(\xi^{-1}(\frac{f}{f_r}))|^{1/2}$ with $\xi^{-1}(\xi(b))=b$ and $\xi'(b)=\frac{d}{db}\xi(b)$. For $\xi(b)=b$, the GC in (9) is Cohen's class in (4), and for $\xi(b)=\ln b$, the GC yields the hyperbolic class [5]. Any generalized warped affine class (GA), $T^{(GA)}_X(t,f)$, is obtained by warping the affine class [12, 14], and can be written in terms of a 2-D kernel $\Phi^{(GA)}_T(b,\beta)$ as

$$T^{(GA)}_X(t,f)=\iint \Phi^{(GA)}_T\Big(\frac{-b}{\xi(\frac{f}{f_r})},\frac{\beta}{\xi(\frac{f}{f_r})}\Big)V_X(b,\beta)e^{j2\pi\frac{t\beta}{\tau(f)}}\frac{db\,d\beta}{|\xi(\frac{f}{f_r})|}. \quad (10)$$

The choice of $\xi(b)$ determines the class to which the GA in (10) simplifies. For $\xi(b)=b$ the GA is the affine class, for $\xi(b)=\xi_\kappa(b)$ it is the κth power class in (7), and for $\xi(b)=e^{\kappa b}$ it is the κth power exponential class given in Section 2.1.

Here, we propose a localized-kernel subclass of the GC and a localized-kernel subclass of the GA. Specifically, we want to group together generalized QTFRs that satisfy the TF concentration property in (1), and provide perfect concentration of the QTFR along given group delay curves. Such QTFRs will necessarily have a kernel with a localized structure as we will show next. For example, the localized GC consists of all generalized warped Cohen's QTFRs whose 2-D kernel in (9) is perfectly localized along a curve $b=F^{(GC)}_T(\beta)$ in the (b,β)-plane,

$$\Phi^{(GC)}_T(b,\beta)=G^{(GC)}_T(\beta)\,\delta(b-F^{(GC)}_T(\beta)), \quad (11)$$

where $F^{(GC)}_T(\beta)\in\mathbb{R}$ and $G^{(GC)}_T(\beta)\geq 0$ are 1-D functions that characterize $T^{(GC)}$. All localized GA QTFRs in (10) have the same localized-kernel structure as in (11), but with (GC) replaced with (GA).

GC and GA QTFRs satisfy the TF concentration property (1) if and only if the conditions listed in Table 1 hold. For example, to determine whether a GC QTFR satisfies property (1) for a given $\lambda(b)$, we first need to check whether a frequency-independent kernel, $F^{(GC)}_T(\beta)$, exists that satisfies condition G–I in Table 1 . If $F^{(GC)}_T(\beta)$ does not exist, then there does not exist any GC QTFR satisfying (1). If a kernel $F^{(GC)}_T(\beta)$ satisfying condition G–I exists, then we substitute it in the frequency-independent ratio in condition G–II to form $G^{(GC)}_T(\beta)$ for a given $r(f)$. Lastly, condition G–III yields the localized-kernel structure in (11).

It is important to note that when the analyzing signal's phase function $\lambda(b)$ in (1) equals the function $\xi(b)$ that specifies the generalized time-shift covariant class in GC (9) (resp. GA (10)), condition G–I always holds for the localized GC (resp. localized GA) in Table 1. Specifically, when $\lambda(b)=\xi(b)$ in Table 1, then $\Lambda^{(GC)}_{f,\beta}(F^{(GC)}_T(\beta))=\beta$ for the GC and $\Lambda^{(GA)}_{f,\beta}(F^{(GA)}_T(\beta))=\xi(\frac{f}{f_r})\beta$ for the GA, and

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Generalized warped classes in (9) or (10)	$r(f)$ in (1)	$G_T^{(\text{class})}(\beta)$ in (11)	$F_T^{(\text{class})}(\beta)$ in (11) for			
			$\lambda(b)=b$	$\lambda(b)=\ln b$	$\lambda(b)=\xi_\kappa(b)$	$\lambda(b)=e^{\kappa b}$
Cohen's (C); $\xi(b)=b$ in (9)	$r_0\, e^{\frac{af}{2f_r}}$	$e^{\frac{a}{f_r}F_T^{(C)}(f_r\beta)}$	arbitrary			$\frac{f_r}{\kappa}\ln[\sinh(\frac{\kappa\beta}{2})/\frac{\kappa\beta}{2}]$
Hyperbolic (H); $\xi(b)=\ln b$ in (9)	$r_0\,\lvert f\rvert^{\alpha}$	$e^{(1+2\alpha)F_T^{(H)}(\beta)}$	$\ln[\sinh(\frac{\beta}{2})/\frac{\beta}{2}]$	arbitrary	$\frac{1}{\kappa}\ln[\sinh(\frac{\kappa\beta}{2})/\frac{\kappa\beta}{2}]$	
Affine (A); $\xi(b)=b$ in (10)	$r_0\,\lvert f\rvert^{\alpha}$	$\lvert(F_T^{(A)}(\beta))^2-\frac{\beta^2}{4}\rvert^{-\alpha}$	arbitrary	$-\frac{\beta}{2}\coth(\frac{\beta}{2})$		
Power (P); $\xi(b)=\xi_\kappa(b)$ in (10)	$r_0\,\lvert f\rvert^{\alpha}$	$\lvert(F_T^{(P)}(\beta))^2-\frac{\beta^2}{4}\rvert^{\frac{\kappa-1-2\alpha}{2\kappa}}$		$-\frac{\beta}{2}\coth(\frac{\beta}{2})$	arbitrary	
Power Exponential (PE); $\xi(b)=e^{\kappa b}$ in (10)	$r_0\, e^{\frac{af}{f_r}}$	$\lvert(F_T^{(PE)}(\beta))^2-\frac{\beta^2}{4}\rvert^{\frac{\kappa-2a}{2\kappa}}$	$-\frac{\beta}{2}\coth(\frac{\beta}{2})$			arbitrary

Table 2: *The 1-D kernels $F_T(\beta)$ and $G_T(\beta)$ for various generalized warped QTFR classes necessary for $T_X(t,f)$ to satisfy the TF concentration property (1) for given signal parameters $r(f)$ and $\lambda(b)$. The choice of $r(f)$ (resp. $\lambda(b)$) only affects the form of $G_T(\beta)$ (resp. $F_T(\beta)$). Blank entries indicate that we did not find an $F_T(\beta)$ to satisfy condition G–I in Table 1.*

condition G–I holds for arbitrary $F_T^{(GC)}(\beta)$ and $F_T^{(GA)}(\beta)$ kernels. This is further demonstrated in Table 2 where we computed the $G_T(\beta)$ and $F_T(\beta)$ kernels for various choices of signal parameters $\lambda(b)$ and $r(f)$ in (1), and for various choices of time shift parameter $\xi(b)$ in (9) and (10). For example, when $\xi(b)=\xi_\kappa(b)=\text{sgn}(b)|b|^\kappa$ in (7), the localized GA is the κth power localized-kernel subclass in Section 3. From Table 2, if $r(f)=r_0|f|^\alpha=f^{-1/2}$ and $\lambda(b)=\ln b$ (such that $X_c(f)=f^{-1/2}e^{-j2\pi c\ln f/f_r}$ in (1) is a hyperbolic chirp), then the kernels simplify to the kernels of the power Bertrand distribution, i.e. $F_T^{(GA)}(\beta)=F_T^{(P)}(\beta)=-\frac{\beta}{2}\coth(\frac{\beta}{2})$ and $G_T^{(GA)}(\beta)=G_T^{(P)}(\beta)=\frac{\beta/2}{\sinh(\beta/2)}$. Thus, the generalization simplifies to known results which state that the power Bertrand distribution is perfectly concentrated along hyperbolae in the TF plane for hyperbolic chirps [5].

The localized-kernel structure in Table 2 is also important as it simplifies the formulation and property kernel constraints for a GC or GA QTFR [13], as its 2-D kernel is now in terms of two 1-D kernels. The localized GC and the localized GA are also important as they provide a unifying framework for many localized-kernel QTFR subclasses. From Tables 1 and 2, the localized GC simplifies to Cohen's localized-kernel subclass in Section 2 when $\xi(b)=b$ and $G_T^{(GC)}(\beta)=G_T^{(C)}(f_r\beta)$, $f_rF_T^{(GC)}(\beta)=F_T^{(C)}(f_r\beta)$, $\Lambda_{f,\beta}^{(GC)}(b)=\Lambda_{f,f_r\beta}^{(C)}(f_rb)$, and $R_{f,\beta}^{(GC)}(b)=R_{f,f_r\beta}^{(C)}(f_rb)$. When $\xi(b)=\ln b$, the localized GC simplifies to the hyperbolic localized-kernel subclass [6]. Similarly, the localized GA simplifies to the affine localized-kernel subclass [9, 10] when $\xi(b)=b$, to the κth power localized-kernel subclass in Section 3 when $\xi(b)=\xi_\kappa(b)$, and to the exponential localized-kernel subclass [13] when $\xi(b)=e^b$.

5. CONCLUSION

The TF concentration property in (1) is an important property as it provides perfectly concentrated analysis of a signal along its group delay function. We showed that when constraining a QTFR to satisfy this property, the QTFR's kernel must necessarily have a localized structure, and we derived this kernel structure for known classes of QTFRs such as Cohen's class and the power classes. We further generalized the localized-kernel subclasses to provide a unifying framework for all generalized-time shift covariant QTFR classes including Cohen's class, the affine class, the hyperbolic class, the κth power class, the exponential class, and the κth power exponential class. As the localized-kernel structure reduces the 2-D kernel of the various classes to two, 1-D kernels, the localized-kernel subclasses have simplified formulations and kernel constraints. The structure can also reduce the QTFR computational complexity and provide more intuitive QTFR design algorithms.

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